Solutions to PS 12 Physics 201

1.

$$\psi_p(x) = \frac{1}{L} e^{\frac{ipx}{\hbar}} \tag{1}$$
$$\psi_p^*(x) = \frac{1}{L} e^{-\frac{ipx}{\hbar}} \tag{2}$$
$$= \psi_{-p}(x) \tag{3}$$

Thus, we have, since $\psi(x)$ is real

$$A_p = \int \psi_p^*(x)\psi(x)dx \tag{4}$$
$$= \int \psi_{-p}(x)\psi(x)dx \tag{5}$$
$$= \left(\int \psi_{-p}^*(x)\psi(x)dx\right)^* \tag{6}$$
$$= A_{-p}^* \tag{7}$$

and hence

$$P(p) = A_p A_p^* \tag{8}$$
$$= A_{-p}^* A_{-p} \tag{9}$$
$$= P(-p) \tag{10}$$

2. For a particle on a ring, we know the lowest energy state is $E_0 = 0$, corresponding to $p = 0$. From the condition that the wavefunction $\psi(x)$ be single valued on the ring, we have that In the first excited state, the particle will have the minimum nonzero momentum satisfying

$$e^{\frac{ipL}{\hbar}} = 1 \tag{11}$$
$$\frac{pL}{\hbar} = 2\pi n \tag{12}$$
$$\tag{13}$$

And thus

$$p_1 = \frac{2\pi\hbar}{L} \tag{14}$$

giving an energy of

$$E_1 = \frac{p_1^2}{2m} \tag{15}$$

$$= \frac{2\pi^2\hbar^2}{mL^2} \tag{16}$$

The frequency of the emitted photon is then

$$f = \frac{E_1 - E_0}{h} \tag{17}$$

$$= \frac{2\pi^2 h}{4\pi^2 mL^2} \tag{18}$$

$$= \frac{h}{2mL^2} \tag{19}$$

$$= 363.7MHz \tag{20}$$

3. (a) The normalized ground state wavefunction is given by

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \tag{21}$$

We note that this, and its square, are symmetric about the point $x = L/2$. Noticing that the function $x - L/2$ is antisymmetric about this same point, we have from symmetry

$$0 = \int_0^L \left(x - \frac{L}{2}\right) |\psi(x)|^2 \, dx \tag{22}$$

$$\frac{L}{2}\int_0^L |\psi(x)|^2 \, dx = \int_0^L x\, |\psi(x)|^2 \, dx \tag{23}$$

$$\langle x \rangle = \frac{L}{2} \tag{24}$$

Next

$$\langle x^2 \rangle = \int_0^L x^2 |\psi(x)|^2 \, dx \tag{25}$$

$$= \frac{2}{L}\int_0^L x^2 \sin^2\left(\frac{\pi x}{L}\right) dx \tag{26}$$

$$= \frac{1}{L}\int_0^L x^2 \left(1 - \cos\left(\frac{2\pi x}{L}\right)\right) dx \tag{27}$$

$$= \frac{1}{L}\left(\frac{L^3}{3} - \int_0^L x^2 \cos\left(\frac{2\pi x}{L}\right) dx\right) \tag{28}$$

$$= \frac{L^2}{3} + \frac{1}{L}\int_0^L \frac{xL}{\pi} \sin\left(\frac{2\pi x}{L}\right) dx \tag{29}$$

$$= \frac{L^2}{3} + \frac{L^2}{2\pi^2} \tag{30}$$

and thus

$$\Delta x = L\sqrt{\frac{1}{12} - \frac{1}{2\pi^2}} \tag{31}$$

(b)

$$\begin{aligned} \langle p \rangle &= \int \psi^*(x)\left(-i\hbar\frac{d\psi(x)}{dx}\right)dx && (32)\\ &= \sum_p \sum_{p'} \int A_p^* \psi_p^*(x) A_{p'}\left(-i\hbar\frac{d\psi_{p'}(x)}{dx}\right)dx && (33) \end{aligned}$$

Now, $\psi_p(x) \propto \exp(ipx/\hbar)$, and thus

$$-i\hbar\frac{d\psi_p(x)}{dx} = p\psi_p(x) \tag{34}$$

Plugging this in, we find

$$\begin{aligned} \langle p \rangle &= \sum_p \sum_{p'} \int A_p^* A_{p'} p' \psi_p^*(x)\psi_{p'}(x)dx && (35)\\ &= \sum_p \sum_{p'} A_p^* A_{p'} p' \delta_{pp'} && (36)\\ &= \sum_p p\left|A_p\right|^2 && (37) \end{aligned}$$

Where we used orthonormality to do the integral over x.

4. (a)

$$\begin{aligned} \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}dx &= -\frac{d}{d\alpha}\int_{-\infty}^{\infty} e^{-\alpha x^2}dx && (38)\\ &= -\frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} && (39)\\ &= \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} && (40) \end{aligned}$$

(b)

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \left|\psi(x)\right|^2 dx && (41)\\ &= A^2\int_{-\infty}^{\infty} e^{-\frac{m\omega x^2}{\hbar}}dx && (42)\\ &= A^2\sqrt{\frac{\pi\hbar}{m\omega}} && (43)\\ A &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} && (44) \end{aligned}$$

(c)

$$\frac{d^2\psi(x)}{dx^2} = A\frac{d}{dx}\left(-\frac{m\omega x}{\hbar}e^{-\frac{m\omega x}{2\hbar}}\right) \tag{45}$$

$$= A\left(-\frac{m\omega}{\hbar} + \frac{m^2\omega^2x^2}{\hbar^2}\right)e^{-\frac{m\omega x^2}{2\hbar}} \tag{46}$$

$$= \left(-\frac{m\omega}{\hbar} + \frac{m^2\omega^2x^2}{\hbar^2}\right)\psi(x) \tag{47}$$

Plugging this in to the eigenvalue equation for E, we find

$$E\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega x^2\psi(x) \tag{48}$$

$$= \left(\frac{\hbar\omega}{2} - \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2x^2\right)\psi(x) \tag{49}$$

$$= \frac{\hbar\omega}{2}\psi(x) \tag{50}$$

and thus

$$E = \frac{\hbar\omega}{2} \tag{51}$$

(d) Using our result from the first part of the problem, we have easily that

$$\langle x^2\rangle = A^2\frac{1}{2\left(\frac{m\omega}{\hbar}\right)}\sqrt{\frac{\pi}{\frac{m\omega}{\hbar}}} \tag{52}$$

$$= \frac{\hbar}{2m\omega} \tag{53}$$

Next, since $\psi(x)$ is even about $x = 0$ while x is odd, we have by symmetry that

$$\langle x\rangle = 0 \tag{54}$$

Thus

$$\Delta x = \sqrt{\langle x^2\rangle} = \sqrt{\frac{\hbar}{2m\omega}} \tag{55}$$

5. Since $E > V_0$, we have that

$$k = \sqrt{\frac{2mE}{\hbar^2}} = 7.25\times 10^{10}\frac{1}{m} \tag{56}$$

and

$$k' = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} = 5.12\times 10^{10}\frac{1}{m} \tag{57}$$

With these values, we find

$$B = \frac{k-k'}{k+k'} = 0.172 \tag{58}$$

$$C = \frac{2k}{k+k'} = 1.17 \tag{59}$$

In the case where $V_0 = 400eV > E$, we still have k given by Eq. (56), however k' now becomes

$$k' = \sqrt{\frac{2m(E - V_0)}{\hbar^2}} = i\kappa \tag{60}$$

Thus we have for B and C

$$B = \frac{k - i\kappa}{k + i\kappa} \tag{61}$$

$$C = \frac{2k}{k + i\kappa} \tag{62}$$

Notice that if we let $z = k + i\kappa$, we can express B as

$$B = \frac{z^*}{z} \tag{63}$$

from which it is evident that

$$|B| = \frac{|z^*|}{|z|} = 1 \tag{64}$$

We know that inside the barrier region,

$$\psi(x) \propto e^{-\kappa x} \tag{65}$$

and thus, the wave function will fall to $1/e$ its initial value at

$$x = \frac{1}{\kappa} = 1.38 \times 10^{-11} m \tag{66}$$

6. (a) Since there is no potential on the ring, we know that energy eigenstates are superpositions of momentum eigenstates with the same magnitude of the momentum. Thus, all we can say knowing the energy is that the particle has momentum

$$p = \pm\sqrt{2mE} \tag{67}$$

Since we do not have enough information to determine the relative odds of either of these values, we cannot compute the probability density.

(b) From the above, the possible values of p are

$$p = \pm\sqrt{2mE} \tag{68}$$

$$= \pm\sqrt{2m\frac{8\pi^2\hbar^2}{mL^2}} \tag{69}$$

$$= \pm\frac{4\pi\hbar}{L} \tag{70}$$

(c) We cannot list the odds of each, because any wave function of the form

$$\psi(x) = Ae^{\frac{4\pi i x}{L}} + Be^{-\frac{4\pi i x}{L}} \tag{71}$$

Is an energy eigenstate of the given energy

(d) If either of the allowed values is measured, the wave function after the measurement will be of the form

$$\psi(x) = \frac{1}{\sqrt{L}} e^{\pm\frac{4\pi i x}{L}} \tag{72}$$

And thus for either value we will have

$$P(x) = |\psi(x)|^2 = \frac{1}{L} \tag{73}$$

7. The wave functions for the $n = 2$ and $n = 3$ states are given by

$$\psi_2(x) = A\sin\left(\frac{2\pi x}{L}\right) \tag{74}$$

$$\psi_3(x) = B\sin\left(\frac{3\pi x}{L}\right) \tag{75}$$

$$\tag{76}$$

Any superposition of these with $|A|^2 = 1/3$ and $|B|^2 = 2/3$ will satisfy the constraint on the odds. Thus, we can choose any of the distinct wavefunctions

$$\psi(x) = \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi x}{L}\right) + \sqrt{\frac{2}{3}}e^{i\phi}\sin\left(\frac{3\pi x}{L}\right) \tag{77}$$

with $0 \leq \phi < 2\pi$. Note that every choice of ϕ in the interval yields a distinct wave function. As an example, we may choose

$$\psi_+(x) = \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi x}{L}\right) + \sqrt{\frac{2}{3}}\sin\left(\frac{3\pi x}{L}\right) \tag{78}$$

and

$$\psi_-(x) = \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi x}{L}\right) - \sqrt{\frac{2}{3}}\sin\left(\frac{3\pi x}{L}\right) \tag{79}$$

8. We have for the eigenvalue equation in the box

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x) \tag{80}$$

which is valid for $0 \leq x \leq L$. We make the ansatz

$$\psi_E(x) = Ae^{ikx} + Be^{-ikx} \tag{81}$$

The constraint that $\psi(0) = 0$ tells us that

$$0 = A + B \tag{82}$$

and so we have

$$\psi_E(x) = A\left(e^{ikx} - e^{-ikx}\right) \tag{83}$$

Next, the constraint $\psi(L) = 0$ tells us

$$e^{ikL} = e^{-ikL} \tag{84}$$

$$e^{2ikL} = 1 \tag{85}$$

$$k = \frac{n\pi}{L} \tag{86}$$

Plugging this in, we find

$$\frac{d^2\psi_E(x)}{dx^2} = -\frac{n^2\pi^2}{L^2} A\left(e^{\frac{n\pi x}{L}} - e^{-\frac{n\pi x}{L}}\right) \tag{87}$$

$$= -\frac{n^2\pi^2}{L^2}\psi_E(x) \tag{88}$$

and by comparing both sides of the eigenvalue equation we arrive at the result

$$E = \frac{n^2\pi^2\hbar^2}{2mL^2} \tag{89}$$

In complete agreement with the result obtained using sin and cos.