PS 12 Physics 201 April 14, 2010 R.Shankar Due April 21.

1. Show that if $\psi(x)$ is real $P(p) = P(-p)$.

2. An electron is in a ring of circumference $L = 1\mu m$. Find the frequency of a photon absorbed when it jumps from the lowest energy state to the one just above it.

3. For a variable V that can take on N values $V_1, V_2, ..V_i, ..V_N$, with probabilities $P(i)$, the average or mean is defined as

$$\langle V \rangle = \sum_i^N P(i) V_i. \tag{1}$$

If the variable is continuous like x, the sum is replaced by an integral. So you should not be surprised if the average of x in a state $\psi(x)$ is defined as

$$\langle x \rangle = \int P(x) x dx = \int \psi^*(x)\psi(x) x dx \tag{2}$$

and the average of x^2 as

$$\langle x^2 \rangle = \int \psi^*(x)\psi(x) x^2 dx. \tag{3}$$

(i) Find $\langle x \rangle$ and $\langle x^2 \rangle$ for a particle of mass m in the ground state of a box of length L. You are encouraged to use symmetry arguments to find $\langle x \rangle$, rather than do integrals.

The technical definition of uncertainty is

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \tag{4}$$

What is Δx for the ground state in a box?

(ii) I claim that in any state $\psi(x)$, the average momentum is

$$\langle p \rangle = \int \psi^*(x) \left(-i\hbar \frac{d\psi(x)}{dx} \right) dx \tag{5}$$

Show that this reduces to

$$\langle p \rangle = \sum_p |A_p|^2 p \tag{6}$$

by writing $\psi(x) = \sum_p A_p \psi_p(x)$ and similarly for $\psi^*(x)$ and putting the two sums into the integral above. (Hint: orthonormality.)

4. HARMONIC OSCILLATOR: VERY IMPORTANT You may assume the following formula

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \tag{7}$$

(i) Differentiate both sides w.r.t α and show that

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} \tag{8}$$

(ii) Consider the function

$$\psi(x) = Ae^{-m\omega x^2/2\hbar}. \tag{9}$$

Choose A to normalize it.

(iii) Consider a harmonic oscillator whose energy in the classical theory is given by

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{10}$$

so that in the quantum version of the oscillator, the wave function for a state of definite energy obeys

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_E(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi_E(x) = E\psi_E(x). \tag{11}$$

Show that the ψ in Eq. 9 satisfies this equation with $E = \frac{\hbar\omega}{2}$.

(iv) Find $\langle x^2 \rangle$ in this state and Δx defined above in Eq. 4.

5. An electron of energy $E = 200eV$ coming in from $x = -\infty$ approaches a barrier of height $V_0 = 100eV$ that starts at $x = 0$ and extends to ∞. Compute the reflection and transmission amplitudes B and C given by

$$B = \frac{k - k'}{k + k'} \qquad C = \frac{2k}{k + k'}. \tag{12}$$

Now consider a barrier $V_0 = 400eV$ and find B and C in terms of k and

$$\kappa = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}.$$

Show that B has modulus 1. We know the wave function falls exponentially in the barrier region now. At what x does ψ drop to $1/e$ of the value at $x = 0$?

6. A particle of mass m is in a ring of circumference L. I catch it in a state of energy $E = 8\pi^2\hbar^2/mL^2$. (i) What is the probability density in this state? Argue that you do not have enough information to answer this and explain why. (ii) What are the possible momenta I can get in this state? (iii) Can you list the the odds for each? (iv) What will be $P(x)$ *after* any one value is measured?

7. Write down two unnormalized, *physically distinct* (i.e., not multiples of each other) wave functions that describe a particle in a box that has 1/3 chance of being in the $n = 2$ state and 2/3 chance of being in the $n = 3$ state.

8. Find the energy functions ψ_E in a box using $e^{\pm ikx}$ instead of $\sin kx$ and $\cos kx$.