Solutions to PS 11 Physics 201

1. (i) The total probabily of the particle being found in the region of $-L/2 \leq x \leq L/2$ should be 1. That is,

$$P(-L/2 \leq x \leq L/2) = \int_{-L/2}^{L/2} |\psi(x)|^2 dx = A^2 a + (A/2)^2 a = \frac{5}{4} A^2 a = 1. \tag{1}$$

Therefore, we have

$$A = \frac{2}{\sqrt{5a}}. \tag{2}$$

(ii) Using the previous result,

$$P(x > 0) = \int_0^{L/2} |\psi(x)|^2 dx = a(\frac{A}{2})^2 = \frac{1}{5}. \tag{3}$$

(iii) Normalized wavefunction with $p = 0$ is given by

$$\psi_0(x) = \frac{1}{\sqrt{L}}. \tag{4}$$

Then, the probablity amplitude of obtaining momentum 0 is

$$A_0 = \int_{-L/2}^{L/2} \psi_0^*(x)\psi(x)dx \tag{5}$$

$$= \int_{-L/2}^{L/2} \frac{1}{\sqrt{L}}\psi(x)dx \tag{6}$$

$$= \frac{3aA}{2\sqrt{L}} \tag{7}$$

$$= 3\sqrt{\frac{a}{5L}}. \tag{8}$$

Therefore,

$$P(p = 0) = \frac{9a}{5L} \tag{9}$$

(iv) The wavefunction after the measurement $\psi'(x)$ is given by the eigenstate associated with the eigenvalue $p = 0$ obtained in the measurement. Therefore,

$$\psi'(x) = \psi_0(x) = \frac{1}{\sqrt{L}}. \tag{10}$$

(v)

$$P(x\rangle 0) = \int_0^{L/2} |\psi'(x)|^2 dx = \frac{1}{2} \tag{11}$$

2. By rewriting the given wavefunction, we have

$$\begin{aligned}
\psi(x) &= 5\cos^2(2\pi x/L) + 2\sin(4\pi x/L) && (12)\\
&= 5\frac{\cos(4\pi x/L)+1}{2} + 2\sin(4\pi x/L) && (13)\\
&= \frac{5}{4}e^{i4\pi x/L} + \frac{5}{4}e^{-i4\pi x/L} + \frac{5}{2} - ie^{i4\pi x/L} + ie^{-i4\pi x/L} && (14)\\
&= (\frac{5}{4} - i)e^{i4\pi x/L} + (\frac{5}{4} + i)e^{-i4\pi x/L} + \frac{5}{2}. && (15)
\end{aligned}$$

Because momentum eigenstates are proportional to $e^{ikx} = e^{ipx/\hbar}$, the possible values of p are $\pm 4\pi\hbar/L$ and 0. Corresponding probabilities for obtaining these values are,

$$\begin{aligned}
P(p = 4\pi\hbar/L) &= \frac{|\frac{5}{4} - i|^2}{|\frac{5}{4} - i|^2 + |\frac{5}{4} + i|^2 + (\frac{5}{2})^2} && (16)\\
&= \frac{41}{182}, && (17)
\end{aligned}$$

$$\begin{aligned}
P(p = -4\pi\hbar/L) &= \frac{|\frac{5}{4} + i|^2}{|\frac{5}{4} - i|^2 + |\frac{5}{4} + i|^2 + (\frac{5}{2})^2} && (18)\\
&= \frac{41}{182}, && (19)
\end{aligned}$$

and

$$\begin{aligned}
P(p = 0) &= \frac{(\frac{5}{2})^2}{|\frac{5}{4} - i|^2 + |\frac{5}{4} + i|^2 + (\frac{5}{2})^2} && (20)\\
&= \frac{50}{91} && (21)
\end{aligned}$$

3. From the normalization condition, we have

$$A = \frac{1}{\sqrt{2a}}. \tag{22}$$

Obviously, $\langle x \rangle = 0$. Also,

$$\langle x^2 \rangle = \int_{-L/2}^{L/2} x^2 |\psi(x)|^2 dx \tag{23}$$

$$= \int_{-a}^{a} x^2 \frac{1}{2a} \tag{24}$$

$$= \frac{a^2}{3}. \tag{25}$$

Therefore,

$$\Delta x = \sqrt{\langle \Delta x^2 \rangle} = \langle x^2 - \langle x \rangle^2 \rangle = \frac{a}{\sqrt{3}}. \tag{26}$$

(Or, you can estimateΔ x simply by the width a.)

Using the fact that normalized wavefunction with momentum p is given by $\psi_p(x) = \frac{1}{\sqrt{L}} e^{ipx/\hbar}$,

$$A_p = \int_{-L/2}^{L/2} \psi_p(x)^* \psi(x) dx \tag{27}$$

$$= \frac{1}{\sqrt{L}} \int_{-a}^{a} e^{-ipx/\hbar} A dx \tag{28}$$

$$= \frac{A}{\sqrt{L}} [\frac{e^{-ipx/\hbar}}{-ip/\hbar}]_{-a}^{a} \tag{29}$$

$$= \frac{1}{\sqrt{2aL}} \frac{2 \sin pa/\hbar}{p/\hbar}. \tag{30}$$

Therefore,

$$|A_p|^2 = \frac{2}{aL} \frac{\sin^2(pa/\hbar)}{(p/\hbar)^2} \tag{31}$$

$$= \frac{2a}{L} \frac{\sin^2(pa/\hbar)}{(pa/\hbar)^2} \tag{32}$$

$$= \frac{2a}{L} \frac{\sin^2 Z}{Z^2}. \tag{33}$$

The first minimum of $\sin^2 Z/Z^2$ occurs at $Z = \pm\pi$ as shown in Fig.1. That is, $p = \pm\pi\hbar/a$.

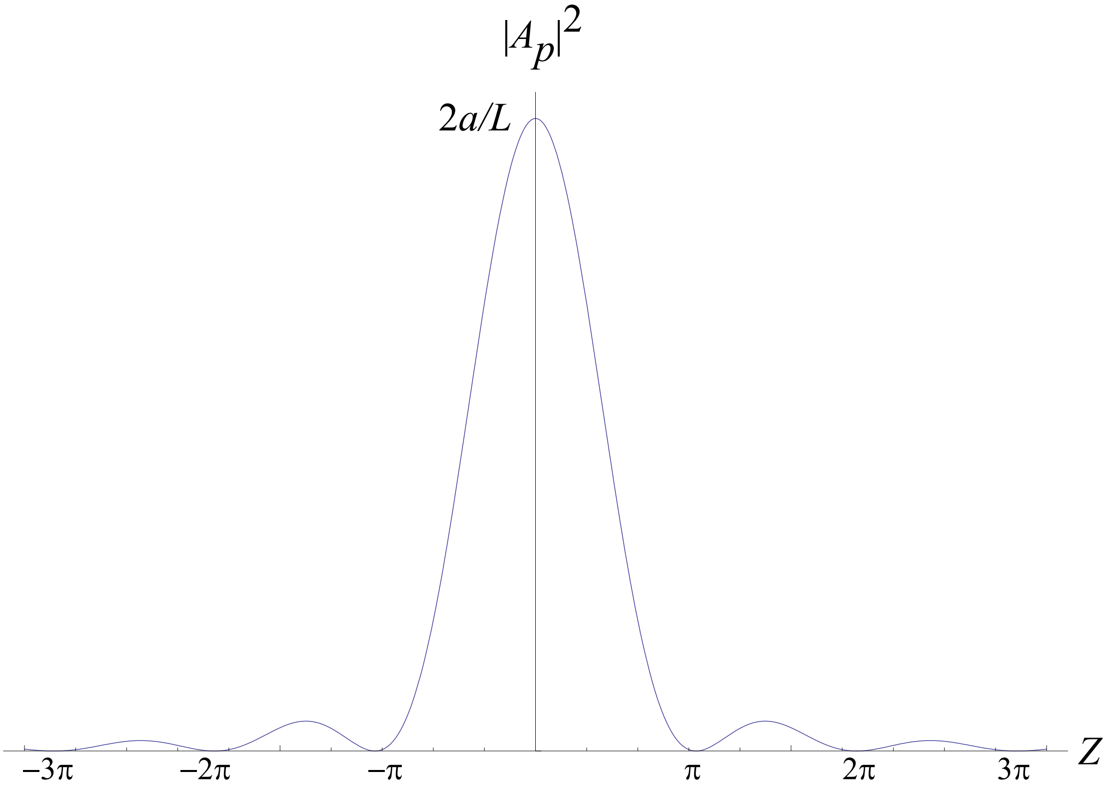

FIG. 1:

If this value is assumed to beΔ p, then

$$\Delta x \Delta p = \frac{\pi\hbar}{a}\frac{a}{\sqrt{3}} = \frac{\pi\hbar}{\sqrt{3}}. \tag{34}$$

Next, we want to show that the sum of probability of obtaining p is 1. Following the steps explained in the problem, we get

$$\sum_p |A_p|^2 = \frac{L}{2\pi\hbar}\int_{-\infty}^{\infty}\frac{2a}{L}\frac{\sin^2 Z}{Z^2}dp \tag{35}$$

$$= \frac{L}{2\pi\hbar}\int_{-\infty}^{\infty}\frac{2a}{L}\frac{\sin^2 Z}{Z^2}\frac{\hbar}{a}dZ \tag{36}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 Z}{Z^2}dZ \tag{37}$$

$$= 1. \tag{38}$$

4.

$$\sum_p |A_p|^2 = \frac{L}{2\pi\hbar}\int \frac{4\alpha^3}{L}(\frac{1}{\alpha^2 + p^2/\hbar^2})^2 dp \tag{39}$$

$$= \frac{2\alpha^3}{\pi\hbar}\int_{-\infty}^{\infty}(\frac{1}{\alpha^2 + p^2/\hbar^2})^2 dp \tag{40}$$

$$= \frac{2\alpha^3}{\pi\hbar}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{\frac{1}{\alpha^2(1+\tan^2 y)}\}^2 \alpha\hbar\frac{dy}{\cos^2 y} \qquad (\alpha\tan y \equiv \frac{p}{\hbar}) \tag{41}$$

$$= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 y \; dy \tag{42}$$

$$= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2y}{2} dy \tag{43}$$

$$= 1 \tag{44}$$