## Solutions to PS 11 Physics 201

1. (i) The total probabily of the particle being found in the region of $-L / 2 \leq x \leq L / 2$ should be 1. That is,

$$
\begin{equation*}
P(-L / 2 \leq x \leq L / 2)=\int_{-L / 2}^{L / 2}|\psi(x)|^{2} d x=A^{2} a+(A / 2)^{2} a=\frac{5}{4} A^{2} a=1 \tag{1}
\end{equation*}
$$

Therefore, we have

$$
\begin{equation*}
A=\frac{2}{\sqrt{5 a}} \tag{2}
\end{equation*}
$$

(ii) Using the previous result,

$$
\begin{equation*}
P(x>0)=\int_{0}^{L / 2}|\psi(x)|^{2} d x=a\left(\frac{A}{2}\right)^{2}=\frac{1}{5} . \tag{3}
\end{equation*}
$$

(iii) Normalized wavefunction with $p=0$ is given by

$$
\begin{equation*}
\psi_{0}(x)=\frac{1}{\sqrt{L}} \tag{4}
\end{equation*}
$$

Then, the probablity amplitude of obtaining momentum 0 is

$$
\begin{align*}
A_{0} & =\int_{-L / 2}^{L / 2} \psi_{0}^{*}(x) \psi(x) d x  \tag{5}\\
& =\int_{-L / 2}^{L / 2} \frac{1}{\sqrt{L}} \psi(x) d x  \tag{6}\\
& =\frac{3 a A}{2 \sqrt{L}}  \tag{7}\\
& =3 \sqrt{\frac{a}{5 L}} \tag{8}
\end{align*}
$$

Therefore,

$$
\begin{equation*}
P(p=0)=\frac{9 a}{5 L} \tag{9}
\end{equation*}
$$

(iv) The wavefunction after the measurement $\psi^{\prime}(x)$ is given by the eigenstate associated with the eigenvalue $p=0$ obtained in the measurement. Therefore,

$$
\begin{equation*}
\psi^{\prime}(x)=\psi_{0}(x)=\frac{1}{\sqrt{L}} . \tag{10}
\end{equation*}
$$

(v)

$$
\begin{equation*}
P(x\rangle 0)=\int_{0}^{L / 2}\left|\psi^{\prime}(x)\right|^{2} d x=\frac{1}{2} \tag{11}
\end{equation*}
$$

## Open Yale courses

© Yale University 2012. Most of the lectures and course material within Open Yale Courses are licensed application of the Creative Commons license.
2. By rewriting the given wavefunction, we have

$$
\begin{align*}
\psi(x) & =5 \cos ^{2}(2 \pi x / L)+2 \sin (4 \pi x / L)  \tag{12}\\
& =5 \frac{\cos (4 \pi x / L)+1}{2}+2 \sin (4 \pi x / L)  \tag{13}\\
& =\frac{5}{4} e^{i 4 \pi x / L}+\frac{5}{4} e^{-i 4 \pi x / L}+\frac{5}{2}-i e^{i 4 \pi x / L}+i e^{-i 4 \pi x / L}  \tag{14}\\
& =\left(\frac{5}{4}-i\right) e^{i 4 \pi x / L}+\left(\frac{5}{4}+i\right) e^{-i 4 \pi x / L}+\frac{5}{2} \tag{15}
\end{align*}
$$

Because momentum eigenstates are proportional to $e^{i k x}=e^{i p x / \hbar}$, the possible values of $p$ are $\pm 4 \pi \hbar / L$ and 0 . Corresponding probabilities for obtaining these values are,

$$
\begin{gather*}
P(p=4 \pi \hbar / L)=\frac{\left|\frac{5}{4}-i\right|^{2}}{\left|\frac{5}{4}-i\right|^{2}+\left|\frac{5}{4}+i\right|^{2}+\left(\frac{5}{2}\right)^{2}}  \tag{16}\\
=\frac{41}{182},  \tag{17}\\
P(p=-4 \pi \hbar / L)=\frac{\left|\frac{5}{4}+i\right|^{2}}{\left|\frac{5}{4}-i\right|^{2}+\left|\frac{5}{4}+i\right|^{2}+\left(\frac{5}{2}\right)^{2}}  \tag{18}\\
=\frac{41}{182} \tag{19}
\end{gather*}
$$

and

$$
\begin{align*}
P(p & =0)=\frac{\left(\frac{5}{2}\right)^{2}}{\left|\frac{5}{4}-i\right|^{2}+\left|\frac{5}{4}+i\right|^{2}+\left(\frac{5}{2}\right)^{2}}  \tag{20}\\
& =\frac{50}{91} \tag{21}
\end{align*}
$$

3. From the normalization condition, we have

$$
\begin{equation*}
A=\frac{1}{\sqrt{2 a}} . \tag{22}
\end{equation*}
$$

Obviously, $\langle x\rangle=0$. Also,

$$
\begin{align*}
\left\langle x^{2}\right\rangle & =\int_{-L / 2}^{L / 2} x^{2}|\psi(x)|^{2} d x  \tag{23}\\
& =\int_{-a}^{a} x^{2} \frac{1}{2 a}  \tag{24}\\
& =\frac{a^{2}}{3} . \tag{25}
\end{align*}
$$

Therefore,

$$
\begin{equation*}
\Delta x=\sqrt{\left\langle\Delta x^{2}\right\rangle}=\left\langle x^{2}-\langle x\rangle^{2}\right\rangle=\frac{a}{\sqrt{3}} . \tag{26}
\end{equation*}
$$

(Or, you can estimate $\Delta x$ simply by the width $a$.)
Using the fact that normalized wavefunction with momentum p is given by $\psi_{p}(x)=$ $\frac{1}{\sqrt{L}} e^{i p x / \hbar}$,

$$
\begin{align*}
A_{p} & =\int_{-L / 2}^{L / 2} \psi_{p}(x)^{*} \psi(x) d x  \tag{27}\\
& =\frac{1}{\sqrt{L}} \int_{-a}^{a} e^{-i p x / \hbar} A d x  \tag{28}\\
& =\frac{A}{\sqrt{L}}\left[\frac{e^{-i p x / \hbar}}{-i p / \hbar}\right]_{-a}^{a}  \tag{29}\\
& =\frac{1}{\sqrt{2 a L}} \frac{2 \sin p a / \hbar}{p / \hbar} . \tag{30}
\end{align*}
$$

Therefore,

$$
\begin{align*}
\left|A_{p}\right|^{2} & =\frac{2}{a L} \frac{\sin ^{2}(p a / \hbar)}{(p / \hbar)^{2}}  \tag{31}\\
& =\frac{2 a}{L} \frac{\sin ^{2}(p a / \hbar)}{(p a / \hbar)^{2}}  \tag{32}\\
& =\frac{2 a}{L} \frac{\sin ^{2} Z}{Z^{2}} . \tag{33}
\end{align*}
$$

The first minimum of $\sin ^{2} Z / Z^{2}$ occurs at $Z= \pm \pi$ as shown in Fig.1. That is, $p=$ $\pm \pi \hbar / a$.

## Open Yale courses



FIG. 1:

If this value is assumed to be $\Delta p$, then

$$
\begin{equation*}
\Delta x \Delta p=\frac{\pi \hbar}{a} \frac{a}{\sqrt{3}}=\frac{\pi \hbar}{\sqrt{3}} . \tag{34}
\end{equation*}
$$

Next, we want to show that the sum of probability of obtaining $p$ is 1 . Following the steps explained in the problem, we get

$$
\begin{align*}
\sum_{p}\left|A_{p}\right|^{2} & =\frac{L}{2 \pi \hbar} \int_{-\infty}^{\infty} \frac{2 a \sin ^{2} Z}{L} \frac{Z^{2}}{} d p  \tag{35}\\
& =\frac{L}{2 \pi \hbar} \int_{-\infty}^{\infty} \frac{2 a}{L} \frac{\sin ^{2} Z}{Z^{2}} \frac{\hbar}{a} d Z  \tag{36}\\
& =\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin ^{2} Z}{Z^{2}} d Z  \tag{37}\\
& =1 \tag{38}
\end{align*}
$$

## Open Yale courses

© Yale University 2012. Most of the lectures and course material within Open Yale Courses are licensed under a Creative Commons Attribution-Noncommercial-Share Alike 3.0 license. Unless explicitly set forth in the applicable Credits section of a lecture, third-party content is not covered under the Creative Commons license. Please consult the Open Yale Courses Terms of Use for limitations and further explanations on the application of the Creative Commons license.
4.

$$
\begin{align*}
\sum_{p}\left|A_{p}\right|^{2} & =\frac{L}{2 \pi \hbar} \int \frac{4 \alpha^{3}}{L}\left(\frac{1}{\alpha^{2}+p^{2} / \hbar^{2}}\right)^{2} d p  \tag{39}\\
& =\frac{2 \alpha^{3}}{\pi \hbar} \int_{-\infty}^{\infty}\left(\frac{1}{\alpha^{2}+p^{2} / \hbar^{2}}\right)^{2} d p  \tag{40}\\
& =\frac{2 \alpha^{3}}{\pi \hbar} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\frac{1}{\alpha^{2}\left(1+\tan ^{2} y\right)}\right\}^{2} \alpha \hbar \frac{d y}{\cos ^{2} y} \quad\left(\alpha \tan y \equiv \frac{p}{\hbar}\right)  \tag{41}\\
& =\frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos ^{2} y d y  \tag{42}\\
& =\frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2 y}{2} d y  \tag{43}\\
& =1 \tag{44}
\end{align*}
$$

