Open Yale courses

Solutions to PS 7 Physics 201

1. The impedance of the circuit is given by

$$
\begin{aligned}
Z(\omega) &= R + \frac{1}{i\omega C} + i\omega L && (1)\\
&= R + i(\omega L - \frac{1}{\omega C}). && (2)
\end{aligned}
$$

Noting the relation between the amplitudes, $|I| = |V|/|Z|$, we have

$$
\frac{|I(\omega)|}{|I_{\max}|} = \frac{|I(\omega)|}{|I(\omega_0)|} = \frac{R}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}. \tag{3}
$$

When $\omega = \omega_0 \pm \delta = \omega_0 \pm R/2L$, provided $\delta/\omega_0 << 1$, we have

$$
\begin{aligned}
\frac{|I(\omega_0 \pm \delta)|}{|I_{\max}|} &= \frac{R}{\sqrt{R^2 + \{(\omega_0 \pm \delta)L - \frac{1}{(\omega_0 \pm \delta)C}\}^2}} && (4)\\
&= \frac{R}{\sqrt{R^2 + \{(\omega_0 \pm \delta)L - \frac{1}{\omega_0 C}(1 \mp \frac{\delta}{\omega_0})\}^2}} && (5)\\
&= \frac{R}{\sqrt{R^2 + (\pm \delta L \pm \frac{1}{\omega_0 C}\frac{\delta}{\omega_0})^2}} && (6)\\
&= \frac{R}{\sqrt{R^2 + (\frac{R}{2} + \frac{R}{2})^2}} && (7)\\
&= \frac{1}{\sqrt{2}}. && (8)
\end{aligned}
$$

2. From the relation $1/Z_{//} = \sum 1/Z_i$, we get

$$
\begin{aligned}
\frac{1}{Z} &= \frac{1}{R} + \frac{1}{\frac{1}{i\omega C}} + \frac{1}{i\omega L} && (9)\\
&= \frac{1}{R} + i\omega C - \frac{i}{\omega L}, && (10)
\end{aligned}
$$

and therefore,

$$
Z = \frac{1}{\frac{1}{R} + i\omega C - \frac{i}{\omega L}} = \frac{R\omega L}{\omega L + i(\omega^2 CL - 1)R}. \tag{11}
$$

3. The impedance of the circuit element shown in the figure satisfies the relation

$$\frac{1}{Z} = \frac{1}{\frac{1}{i\omega C}} + \frac{1}{R + i\omega L} \tag{12}$$

$$= i\omega C + \frac{1}{R + i\omega L} \tag{13}$$

$$= i\omega C + \frac{R - i\omega L}{R^2 + \omega^2 L^2} \tag{14}$$

$$= \frac{R + i\{(R^2 + \omega^2 L^2)\omega C - \omega L\}}{R^2 + \omega^2 L^2}. \tag{15}$$

$$\tag{16}$$

Noting that Im$[Z]$ =0 (Z is real.) $\Leftrightarrow$ Im$[1/Z]$ =0, we have

$$\mathrm{Im}[Z] = 0 \Leftrightarrow (R^2 + \omega^2 L^2)\omega C - \omega L = 0 \tag{17}$$

$$\Leftrightarrow (R^2 + \omega^2 L^2)C - L = 0, \text{or } \omega = 0 \tag{18}$$

$$\Leftrightarrow \omega = 0, \sqrt{\frac{L - CR^2}{CL^2}}. \tag{19}$$

Of course, $\sqrt{\frac{L-CR^2}{CL^2}}$ is real only if $L > CR^2$. Otherwise, the impedance is real only for $\omega = 0$ (Note that $Z = \infty$ for $\omega = 0$).

4. As seen in problem 1, the impedance is given by

$$Z(\omega) = R + \frac{1}{i\omega C} + i\omega L \tag{20}$$

$$= R + i(\omega L - \frac{1}{\omega C}). \tag{21}$$

Clearly, R_1 =100 Ω gives the minimum impedance, and R_2 =200 Ω gives the maximum impedance. Next, we have to consider the imaginary part of the impedance. For $\omega = 2000$, we get

$$\omega L_1 - \frac{1}{\omega C_1} = 2000\ \mathrm{s}^{-1} \times 1\ \mathrm{mH} - \frac{1}{2000\ \mathrm{s}^{-1} \times 1\ \mu\mathrm{F}} = -498\Omega\ , \tag{22}$$

$$\omega L_1 - \frac{1}{\omega C_2} = 2000\ \mathrm{s}^{-1} \times 1\ \mathrm{mH} - \frac{1}{2000\ \mathrm{s}^{-1} \times 100\ \mu\mathrm{F}} = -3\ \Omega, \tag{23}$$

$$\omega L_2 - \frac{1}{\omega C_1} = 2000\ \mathrm{s}^{-1} \times 2\ \mathrm{mH} - \frac{1}{2000\ \mathrm{s}^{-1} \times 1\ \mu\mathrm{F}} = -496\Omega\ , \tag{24}$$

and

$$\omega L_2 - \frac{1}{\omega C_2} = 2000\ \mathrm{s}^{-1} \times 2\ \mathrm{mH} - \frac{1}{2000\ \mathrm{s}^{-1} \times 100\ \mu\mathrm{F}} = -1\ \Omega. \tag{25}$$

Therefore, (R_1, C_2, L_2) gives the minimum impedance $|Z_{\text{min}}| = \sqrt{100^2 + 1^2} \approx 100\ \Omega$, and (R_2, C_1, L_1) gives the maximum impedance $|Z_{\text{max}}| = \sqrt{200^2 + 498^2} \approx 537\ \Omega$.

5. The impedance Z_2 at $\omega = 500$ is given by

$$Z_2(\omega = 500) = 15\ \Omega + \frac{1}{i \times 500\ \mathrm{s}^{-1} \times 2\ \mu\mathrm{F}} = (15 - 1000i)\ \Omega \approx 1000.1\ e^{-1.556i}\ \Omega, \tag{26}$$

and the total impedance is

$$Z_{\text{tot}}(\omega = 500) = (25 - 1000i)\ \Omega \approx 1000.3\ e^{-1.545i}\ \Omega. \tag{27}$$

Using these, we can calculate the power loss across Z_2. However, we have to note that $P_2 = I_2 V_2 = \mathrm{Re}[\tilde{I}_2]\mathrm{Re}[\tilde{V}_2] \neq Re[\tilde{I}_2 \tilde{V}_2]$, where $\tilde{A}$ is the imaginary expression of A. (Operations such as derivative or integration commute with an operation of taking Re[], that is, the order of operations does not matter. Actually, this fact makes use of complex number convenient for this kind of problems. However, multiplication does not commute with Re[]. Also note that complex numbers are "imaginary" tool to make calculation easier and that physical quantities we can observe in experiments are always real.) Therefore,

$$\begin{aligned}
P_2 \equiv I_2 V_2 = I(IZ_2) &= \left(\frac{V}{Z_{\text{tot}}}\right)\frac{VZ_2}{Z_{\text{tot}}} && (28)\\
&= \mathrm{Re}\left[\frac{30e^{i500t}[\mathrm{V}]}{1000.3\ e^{-1.545i}\ \Omega}\right] \mathrm{Re}\left[\frac{30e^{i500t}[\mathrm{V}](1000.1\ e^{-1.556i}\ \Omega)}{1000.3\ e^{-1.545i}\ \Omega}\right] && (29)\\
&= 0.900\ \mathrm{Re}[e^{i(500t+1.545)}]\ \mathrm{Re}[e^{i(500t-0.011)}]\ [\mathrm{W}] && (30)\\
&= 0.900\ \cos(500t + 1.545)\ \cos(500t - 0.011)\ [\mathrm{W}] && (31)\\
&= 0.450\ \{\cos(1000t + 1.534) + \cos 1.556\}\ [\mathrm{W}] && (32)\\
& && (33)
\end{aligned}$$

Also from this, we can easily calculate

$$(\text{Time average of power loss}) = 0.450\ \cos 1.556 = 6.66\ \mathrm{mW}. \tag{34}$$

6. The electric field between the plates is

$$\mathbf{E}(r) = \begin{cases} \frac{V(t)}{d} & (r < a) \\ 0 & (r > a), \end{cases} \tag{35}$$

where $d = 2$ cm is the separation between the plates and $a = 4$ cm is a radius of the plates. Noting that the capacitance has rotation symmetry about the central axis, we have from Maxwell equation,

$$\oint \mathbf{B} \cdot d\mathbf{l} = 2\pi r B_\theta(r) = \epsilon_0 \mu_0 \int d\mathbf{S} \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2} \int d\mathbf{S} \frac{\partial \mathbf{E}}{\partial t}. \tag{36}$$

Therefore,

$$\mathbf{B}(r) = \begin{cases} \frac{r}{2c^2 d} \frac{dV(t)}{dt} \mathbf{e}_\theta & (r < a) \\ \frac{a^2}{2c^2 r d} \frac{dV(t)}{dt} \mathbf{e}_\theta & (r > a), \end{cases} \tag{37}$$

where $\frac{dV(t)}{dt} = (-200\pi \times 200 \sin 200\pi t)$V/s, whose amplitude is 40000π V/s. B reaches its maximum at $r = a$. With actual numbers plugged in,

$$|B_{\mathrm{max}}| = \frac{a}{2c^2 d} \left| \frac{dV(t)}{dt} \right| \tag{38}$$

$$= \frac{2\mathrm{cm}}{2 \times (3 \times 10^8 \mathrm{m/s})^2 \times 4\mathrm{cm}} \times 40000\pi \mathrm{V/s} \tag{39}$$

$$= 1.11 \times 10^{-13}\ \mathrm{T}. \tag{40}$$