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Solutions to PS 2 Physics 201

1.

$$\begin{aligned}
\mathbf{E} &= \mathbf{i}\int \frac{k_e dq}{r^2} && (1)\\
&= \mathbf{i}\int_{-L}^{L} \frac{k_e \lambda_0 x dx}{L(x_0 - x)^2} && (2)\\
&= \mathbf{i}\frac{k_e\lambda_0}{L}\int_{-L}^{L} \frac{(x-x_0)dx}{(x-x_0)^2} + \frac{x_0 dx}{(x-x_0)^2} && (3)\\
&= \mathbf{i}\frac{k_e\lambda_0}{L}\left[\ln\left(\frac{x_0 - L}{x_0 + L}\right) + \frac{x_0}{x_0 - L} - \frac{x_0}{x_0 + L}\right] && (4)\\
&= \mathbf{i}\frac{k_e\lambda_0}{L}\left[\ln\left(\frac{x_0 - L}{x_0 + L}\right) + \frac{2x_0 L}{x_0^2 - L^2}\right] && (5)
\end{aligned}$$

To find the field for $x_0 \to \infty$, we first want to rewrite this result in terms of the small parameter $\frac{L}{x_0}$. Doing so yields

$$\mathbf{E} = \mathbf{i}\frac{k_e\lambda_0}{L}\left[\ln\left(\frac{1 - \frac{L}{x_0}}{1 + \frac{L}{x_0}}\right) + \frac{2\frac{L}{x_0}}{1 - \left(\frac{L}{x_0}\right)^2}\right] \quad (6)$$

Next, we perform a Taylor expansion in terms of $\frac{L}{x_0}$ about the point $\frac{L}{x_0} = 0$. For the logarithm term, we find

$$\ln\left(\frac{1 - \frac{L}{x_0}}{1 + \frac{L}{x_0}}\right) \approx 0 - \frac{2L}{x_0} - \frac{2L^3}{3x_0^3} \quad (7)$$

and for the second term

$$\begin{aligned}
\frac{2x_0 L}{x_0^2 - L^2} &= 2\frac{L}{x_0}\left[\frac{1}{1 - \left(\frac{L}{x_0}\right)^2}\right] && (8)\\
&= 2\frac{L}{x_0}\left[1 + \left(\frac{L}{x_0}\right)^3\right] && (9)\\
&= \frac{2L}{x_0} + \frac{2L^3}{x_0^3} && (10)
\end{aligned}$$

Putting this all together, we arrive at a final approximation for $\mathbf{E}$ given by

$$\mathbf{E} = \mathbf{i}\frac{k_e\lambda_0}{L}\left(-\frac{2L}{x_0} - \frac{2L^3}{3x_0^3} + \frac{2L}{x_0} + \frac{2L^3}{x_0^3}\right) \tag{11}$$

$$= \mathbf{i}\frac{k_e\lambda_0}{L}\frac{4L^3}{3x_0^3} \tag{12}$$

$$= \mathbf{i}\frac{\lambda_0 L^2}{3\pi\epsilon_0 x_0^3} \tag{13}$$

as desired. Comparing this to the expression for a dipole field aligned with the axis of a dipole, we find

$$\mathbf{i}\frac{2p}{4\pi\epsilon_0 x_0^3} = \mathbf{i}\frac{\lambda_0 L^2}{3\pi\epsilon_0 x_0^3} \tag{14}$$

$$p = \frac{2\lambda_0 L^2}{3} \tag{15}$$

2.

$$\tau = \mathbf{p} \times \mathbf{E} \tag{16}$$

$$= -pE\sin\theta\mathbf{k} \tag{17}$$

$$= \mathbf{k}(10^{-29})(0.5)\sin\frac{\pi}{6}N\cdot m \tag{18}$$

$$= \mathbf{k}2.5\times 10^{-30}N\cdot m \tag{19}$$

To find the work done, we use

$$W_{done} = -W_e = \Delta U \tag{20}$$

$$= U(\pi) - U\left(\frac{\pi}{6}\right) \tag{21}$$

$$= -pE\cos\pi + pE\cos\frac{\pi}{6} \tag{22}$$

$$= pE\left(1 + \frac{\sqrt{3}}{2}\right) \tag{23}$$

$$= 5\left(1 + \frac{\sqrt{3}}{2}\right)\times 10^{-30}N\cdot m \tag{24}$$

Finally, for the frequency of small oscillations, we use Newton's second law

$$\tau = \frac{dL}{dt} \tag{25}$$

$$-pE\sin\theta = I\ddot{\theta} \tag{26}$$

$$\tag{27}$$

expanding $\sin\theta$ for small θ, we find

$$-\frac{pE}{I}\theta = \ddot{\theta} \tag{28}$$

$$\omega^2 = \frac{pE}{I} \tag{29}$$

$$\tag{30}$$

Plugging in the numbers, we have

$$I = m\left(\frac{d}{2}\right)^2 + m\left(\frac{d}{2}\right)^2 \tag{31}$$

$$= \frac{md^2}{2} \tag{32}$$

$$= 5 \times 10^{-48} kg \cdot m^2 \tag{33}$$

and thus

$$\omega = \sqrt{\frac{pE}{I}} \tag{34}$$

$$= \sqrt{\frac{5 \times 10^{-30}}{5 \times 10^{-48}}}\frac{rad}{s} \tag{35}$$

$$= 10^9 \frac{rad}{s} \tag{36}$$

3. By spherical symmetry, we know automatically that the electric field everywhere will be purely in the radial direction. Using this fact, we can apply Gauss' law by finding the flux through a sphere of radius r centered about the origin,

$$\int \mathbf{E} \cdot d\mathbf{A} = 4\pi r^2 E_r = \frac{Q_{enclosed}}{\epsilon_0} \tag{37}$$

For $r < a$, we have that

$$Q_{enclosed} = -\frac{\frac{4}{3}\pi r^3 Q}{\frac{4}{3}\pi a^3} \tag{38}$$

$$= -Q\frac{r^3}{a^3} \tag{39}$$

and therefore

$$\mathbf{E}(r < a) = -\mathbf{e}_r \frac{k_e Q r}{a^3} \tag{40}$$

For $a \leq r < b$, we have that $Q_{enclosed} = -Q$, and so

$$\mathbf{E}(a \leq r < b) = -\mathbf{e}_r \frac{k_e Q}{r^2} \tag{41}$$

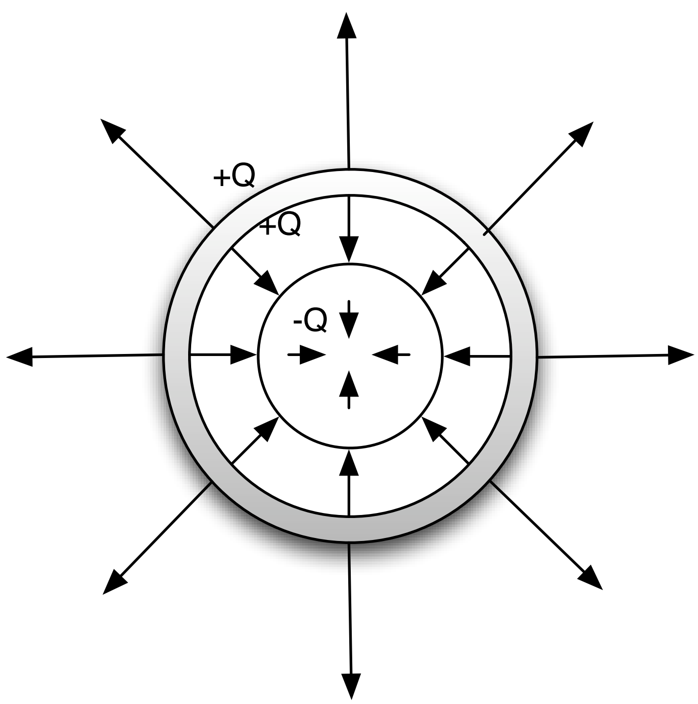
For $b \leq r < c$, the field must be zero since this describes the interior of a conductor, meaning that a charge of $+Q$ must reside on the interior surface of the conducting shell. Therefore

$$\mathbf{E}(b \leq r < c) = 0 \tag{42}$$

Finally, for $r \geq c$, it must be that $Q_{enclosed} = +Q$, and therefore

$$\mathbf{E}(r \geq c) = \mathbf{e}_r \frac{k_e Q}{r^2} \tag{43}$$

Below is a sketch showing where the charges reside, and some field lines.



4. Exploiting the cylindrical symmetry of the problem tells us that the field directed radially (i.e. in the $\mathbf{e}_r$ direction) away from the axis of the cylinders, and that

$$2\pi r L E_r = \frac{Q_{enclosed}}{\epsilon_0} \tag{44}$$

where L is the length of our cylindrical Gaussian surface.

Thus, since the cylinders are hollow, we know that there is no charge enclosed for $r < a$, and thus

$$\mathbf{E}(r < a) = 0 \tag{45}$$

For $a \leq r < b$, we have that

$$Q_{enclosed} = \lambda L \tag{46}$$

and therefore

$$\mathbf{E}(a \leq r < b) = \mathbf{e}_r \frac{2k_e\lambda}{r} \tag{47}$$

Lastly, for $r \geq b$, we have that $Q_{enclosed} = 0$, so

$$\mathbf{E}(r \geq b) = 0 \tag{48}$$

To find the surface charge density σ on the inner cylinder, we note that we can express the total charge on a length L of the cylinder as either

$$Q = 2\pi a L \sigma \tag{49}$$

or as

$$Q = \lambda L \tag{50}$$

Equating these two expressions, we find that

$$\sigma = \frac{\lambda}{2\pi a} \tag{51}$$

We can substitute this result into our expression for the field between the cylinder to find

$$\begin{aligned} \mathbf{E}(a < r < b) &= \mathbf{e}_r \frac{4\pi k_e a\sigma}{r} &(52)\\ &= \mathbf{e}_r \frac{a\sigma}{\epsilon_0 r} &(53) \end{aligned}$$

For $b - a \ll a$, we have that between the cylinders $r - a \ll a$, and thus

$$\begin{aligned} \mathbf{E}(a < r < b) &= \mathbf{e}_r \frac{a\sigma}{\epsilon_0 r} &(54)\\ &= \mathbf{e}_r \frac{a\sigma}{\epsilon_0(a + r - a)} &(55)\\ &\approx \mathbf{e}_r \frac{\sigma}{\epsilon_0} &(56) \end{aligned}$$

This is equal in magnitude to the field of a parallel plate capacitor of the same charge density. Furthermore, on a very small scale $\mathbf{e}_r$ does not vary significantly with the polar angle, and thus may be approximated as a cartesian unit vector. Thus, this setup locally approximates a parallel plate capacitor

5. By Gauss' law,

$$\Phi_e = \frac{1C}{\epsilon_0} \tag{57}$$

Thus, by the symmetry of the cube, we must have that the flux through one of the faces is given by

$$\Phi_e = \frac{1C}{6\epsilon_0} = 1.88 \times 10^{10} \frac{N \cdot m^2}{C} \tag{58}$$

6. For $r < R$ we have that

$$\begin{aligned} Q_{enclosed} &= \int_0^r \int_0^\pi \int_0^{2\pi} \rho(r) r^2 \sin\theta d\phi d\theta d\rho & (59) \\ &= 4\pi A \int_0^r r^4 dr & (60) \\ &= \frac{4\pi A}{5} r^5 & (61) \end{aligned}$$

Evaluating this same expression at $r = R$ gives that the total charge Q is

$$Q = \frac{4\pi A}{5} R^5 \tag{62}$$

and thus for $r < R$

$$Q_{enclosed} = Q \frac{r^5}{R^5} \tag{63}$$

Thus, Gauss' law tells us that

$$\mathbf{E}(r < R) = \mathbf{e}_r \frac{k_e Q r^3}{R^5} \tag{64}$$

For $r > R$, we have that $Q_{enclosed} = Q$, and therefore

$$\mathbf{E}(r \geq R) = \mathbf{e}_r \frac{k_e Q}{r^2} \tag{65}$$

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7. By the Pythagorean theorem, the radius of each disc as a function of z is given by

$$r(z) = \sqrt{R^2 - z^2} \tag{66}$$

and thus the area A of each disc is

$$A(z) = \pi \left(R^2 - z^2\right) \tag{67}$$

The volume of the sphere is then given by integrating over all discs contained in the sphere, i.e. from $z = -R$ to $z = R$. This yields

$$V = \pi \int_{-R}^{R} \left(R^2 - z^2\right) dz \tag{68}$$
$$= \pi \left(2R^3 - \frac{2R^3}{3}\right) \tag{69}$$
$$= \frac{4}{3}\pi R^3 \tag{70}$$

as desired

8. Knowing that Gauss' law follows from Coulomb's law, we can define an analogue of Gauss's law for the gravitational field $\mathbf{G}$. Examining the form of both Newton's and Coulomb's law, we have

$$\mathbf{G} = -\frac{GM}{r^2}\mathbf{e}_r \tag{71}$$

and

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\mathbf{e}_r \tag{72}$$

where the fields $\mathbf{E}$ and $\mathbf{G}$ are the forces on a unit charge (unit mass) due to charge q (mass M). By comparison, we see that M plays the same roll as q, and likewise $-G$ plays the same roll as $\frac{1}{4\pi\epsilon_0}$. From this we arrive at Gauss's law for gravitation,

$$\int \mathbf{G} \cdot d\mathbf{A} = -4\pi G M_{enclosed} \tag{73}$$

9. Using Gauss' law, we have that for $r = .5m$, $Q_{enclosed} = 1\mu C$, and thus

$$\mathbf{E} = \hat{r}\frac{k_e(1\mu C)}{(.5m)^2} = 3.6 \times 10^4 \frac{N}{C}\mathbf{e}_r \tag{74}$$

Next, for $r = 2m$, $Q_{enclosed} = -1\mu C$, and thus

$$\mathbf{E} = -\hat{r}\frac{k_e(1\mu C)}{(2m)^2} = -2.2 \times 10^3 \frac{N}{C}\mathbf{e}_r \tag{75}$$

10. We can examine this situation as a solid sphere of uniform charge density ρ and radius R superimposed with a solid sphere of uniform charge density $-\rho$ with radius $\frac{R}{2}$. Let $\mathbf{r}_1$ denote the vector from the center of the larger sphere to a point within the smaller sphere, and let $\mathbf{r}_2$ denote the vector from the center of the smaller sphere to that same point.

First, we use Gauss' law to find the field $\mathbf{E}_+$ due to the larger sphere. At a distance r_1, we have that the charge enclosed is given by

$$Q_{enclosed} = \frac{4}{3}\pi r_1^3 \rho \tag{76}$$

and thus the field $\mathbf{E}_+$ is given by

$$\mathbf{E}_+ = \frac{\rho r_1}{3\epsilon_0}\mathbf{e}_{r_1} = \frac{\rho}{3\epsilon_0}\mathbf{r}_1 \tag{77}$$

Similarly, for the field E_- due to the smaller, negatively charged sphere, we find

$$\mathbf{E}_- = -\frac{\rho r_2}{3\epsilon_0}\mathbf{e}_{r_2} = -\frac{\rho}{3\epsilon_0}\mathbf{r}_2 \tag{78}$$

Summing together these two contributions to find the total field in the cavity, we get

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho}{3\epsilon_0}\left(\mathbf{r}_1 - \mathbf{r}_2\right) \tag{79}$$

But from the figure, we can see that

$$\mathbf{r}_1 - \mathbf{r}_2 = \frac{R}{2}\hat{\mathbf{x}} \tag{80}$$

Thus,

$$\mathbf{E} = \frac{\rho R}{6\epsilon_0}\hat{\mathbf{x}} \tag{81}$$

which describes a uniform field in the $\hat{\mathbf{x}}$ direction