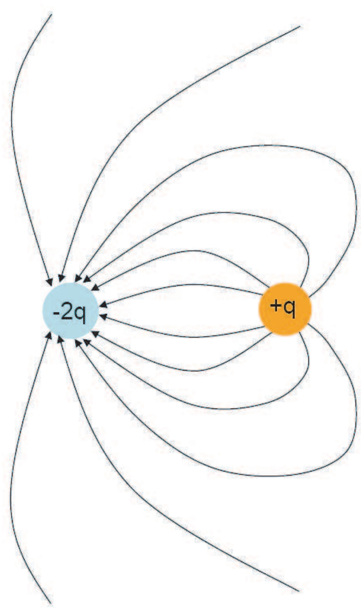
Solutions to Problem Set 1 Physics 201b January 13, 2010.

1. (i) To produce a sphere with $\frac{1}{8}$ μC you need to first let the two spheres touch each other so that the charges redistribute equally on the two spheres. Now each sphere has a charge of $\frac{1}{2}$ μC and grounding one of the spheres discharges it again. Repeating this two more times leaves you with the two spheres charged with $\frac{1}{8}$ μC each.

   (ii) If you cannot ground anything you will need three initially discharged spheres to get $\frac{1}{8}$ μC. Every time you let the charged sphere touch a discharged sphere, its charge reduces to half the value.

   (iii) To get $\frac{5}{16}$ μC, you simply need to let the sphere with $\frac{1}{8}$ μC touch the sphere with $\frac{1}{2}$ μC (the one that first touched the fully charged sphere). This will get you a charge of $\frac{1}{2} \times (\frac{1}{8}\ \mu C\ +\ \frac{1}{2}\ \mu C)\ =\ \frac{5}{16}\ \mu C$.

2. See figure.



3. The symmetry of the problem tells us immediately that the force on the proton is zero. For each electron that attracts the proton there is an electron on the opposite side canceling out the force of the first. Removing any one electron has the effect that one electron is without a pair and thus the resulting force on the proton (in the direction of the electron) simply is

$$\begin{aligned} F &= \frac{e^2}{4\pi\varepsilon_0 r^2} \\ &= 2.3 \times 10^{-28}\ N\ . \end{aligned} \tag{1}$$

4. (i) Let

$$f(x) = (1+x)^p \quad \text{and} \quad x_0 = 0. \tag{2}$$

We then have

$$\left.\frac{df}{dx}\right|_{x_0} = p(1+x_0)^{p-1} = p \tag{3}$$

$$\left.\frac{d^2f}{dx^2}\right|_{x_0} = p(p-1)(1+x_0)^{p-2} = p(p-1) \tag{4}$$

$$\left.\frac{d^3f}{dx^3}\right|_{x_0} = p(p-1)(p-2)(1+x_0)^{p-3} = p(p-1)(p-2), \tag{5}$$

and finally

$$\begin{aligned} f(x) &= f(0) + \left.\frac{df}{dx}\right|_{x=0} x + \frac{1}{2!}\left.\frac{d^2f}{dx^2}\right|_{x=0} x^2 + \frac{1}{3!}\left.\frac{d^3f}{dx^3}\right|_{x=0} x^3 + ... \\ \Longrightarrow \quad f(x) &= 1 + px + \frac{p(p-1)x^2}{2!} + \frac{p(p-1)(p-2)x^3}{3!} + ... \end{aligned} \tag{6}$$

(ii) Let

$$g(x) = \ln(1+x) \quad \text{and} \quad x_0 = 0. \tag{7}$$

Then

$$\left.\frac{dg}{dx}\right|_{x_0} = \frac{1}{1+x_0} = 1 \tag{8}$$

$$\left.\frac{d^2g}{dx^2}\right|_{x_0} = \frac{-1}{(1+x_0)^2} = -1 \tag{9}$$

$$\left.\frac{d^3g}{dx^3}\right|_{x_0} = \frac{2}{(1+x_0)^3} = 2\,. \tag{10}$$

We finally find

$$\begin{aligned} g(x) &= g(0) + \left.\frac{dg}{dx}\right|_{x=0} x + \frac{1}{2!}\left.\frac{d^2g}{dx^2}\right|_{x=0} x^2 + \frac{1}{3!}\left.\frac{d^3g}{dx^3}\right|_{x=0} x^3 + ... \\ \Longrightarrow \quad g(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + ... \end{aligned} \tag{11}$$

5. (1)Let'us think about the balance of the forces to the left particle. Noting the change in the length of each spring is $a/2$, we get

$$F_{tot} = -k\ (-\frac{a}{2}) - \frac{q^2}{4\pi\varepsilon_0(2a)^2} = 0. \tag{12}$$

Solving for k, we get

$$k = \frac{q^2}{8\pi\varepsilon_0 a^3}. \tag{13}$$

(2)In the same way, noting that the change in the length of each spring is $a/4$, we get the balance equation for the left particle.

$$F_{tot} = -k\ (\frac{a}{4}) + \frac{q^2}{4\pi\varepsilon_0(\frac{a}{2})^2} = 0. \tag{14}$$

Solving for k, we get

$$k = \frac{4q^2}{\pi\varepsilon_0 a^3}. \tag{15}$$

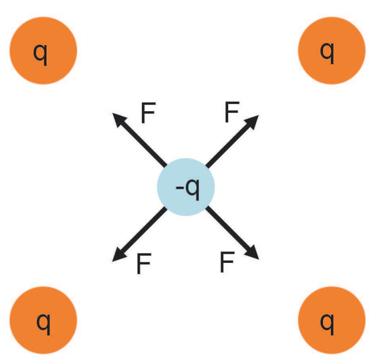
(3) By applying the result of the problem 4, and using the result from part (1), we get

$$\begin{aligned}
F_{tot} &= -k(-a/2+x) - \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2a-x)^2} &(16)\\
&= ka/2 - kx - \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4a^2}\frac{1}{(1-\frac{x}{2a})^2} &(17)\\
&= ka/2 - kx - \frac{q^2}{16\pi\varepsilon_0 a^2}(1+2\frac{x}{2a}+\cdots) &(18)\\
&= -(k+\frac{q^2}{16\pi\varepsilon_0 a^3})x &(19)\\
&\equiv -k_{eff}x. &(20)
\end{aligned}$$

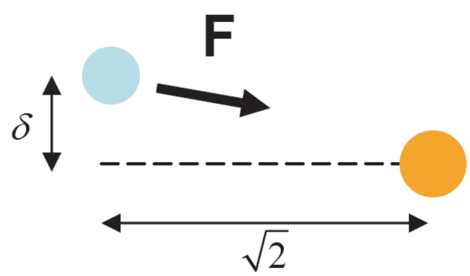
where k_{eff} is the effective force constant in the presence of the electric force (we use k_{eff} to avoid confusion with the electric constant k_e). From the relation $\omega = \sqrt{k_{eff}/m}$, we get

$$\omega = \sqrt{(k+\frac{q^2}{16\pi\varepsilon_0 a^3})/m} = \sqrt{\frac{3q^2}{16m\pi\varepsilon_0 a^3}} \tag{21}$$

6. (i) The force of each unit charge q at the corners exerted on the charge $-q$ in the center is $|\mathbf{F}| = k_e\frac{q^2}{r^2}$, where r is the distance between the center charge and the corner charge (here $r = \sqrt{2}$). As seen in the picture below, forces from charges at opposite corners cancel each other, resulting in a zero net force.



(ii) Let's write out explicitly the force vectors for this case. As always, according to Coulomb's law, the force from the corner charge i pointing towards the center is $\mathbf{F}_i = F\,\hat{\mathbf{r}}_i$ where $F = k_e \frac{-q^2}{r^2}$ and $\hat{\mathbf{r}}_i$ is the unit vector pointing from the charge i towards the center charge.



For convenience, let the origin be at the position of the center charge and let the corner charges have coordinates (x, y, z). Then $\hat{\mathbf{r}}_i = \frac{1}{r}(x\,\mathbf{i}+y\mathbf{j}+z\mathbf{k})$, where $r = \sqrt{x^2+y^2+z^2}$. So when we displace the center charge by an amout δ in the $+z$ direction, then $\hat{\mathbf{r}}_i = \frac{1}{r}(\pm 1\,\mathbf{i} + \pm 1\mathbf{j} + \delta\mathbf{k})$ and $r = \sqrt{2+\delta^2}$ and the sign $\pm$ depends on which charge we consider. From symmetry considerations we immediately see that the resulting force will have a component in the z-direction only.

$$\begin{aligned} \sum_{i=1}^{4} \mathbf{F}_i &= 4 \cdot k_e \frac{-q^2}{r^3} \delta\,\mathbf{k} = 4 \cdot k_e \frac{-q^2}{(2+\delta^2)^{3/2}} \delta\,\mathbf{k} \\ &= 4 \cdot k_e \frac{-q^2}{2^{3/2}} (1 - 3/4\delta^2 + \ldots) \delta\,\mathbf{k} \end{aligned} \tag{23}$$

where for the last equality we have used the Taylor expansion found in problem 2. Neglecting all terms of order 2 or higher in δ we get

$$\mathbf{F} = \sum_{i=1}^{4} \mathbf{F}_i = -(\sqrt{2}\,k_e\,q^2)\,\delta\,\mathbf{k}\,. \tag{24}$$

So we finally find the spring constant k to be

$$k = \sqrt{2}\,k_e\,q^2\,. \tag{25}$$

(iii) From classical mechanics we know that the angular frequency of a spring-mass system is simply $\omega = \sqrt{\frac{k}{m}}$, so here

$$\omega = \sqrt{\frac{\sqrt{2}\, k_e\, q^2}{m}}\,. \tag{26}$$

(iv) To determine the speed with which the charge will cross the origin, we use energy conservation:

$$\begin{aligned} E_{kin} &= E_{pot} \\ \frac{1}{2}mv^2 &= \frac{1}{2}kz^2 = \frac{1}{2}\sqrt{2}\, k_e\, q^2\delta^2. \end{aligned} \tag{27}$$

Thus

$$v = \sqrt{\frac{\sqrt{2}\, k_e\, q^2}{m}}\delta = \omega\, \delta. \tag{28}$$

(v) Using the same approach as in part (ii) we can write out the forces explicitly as:

$$\begin{aligned} \mathbf{F}_1 &= k_e\frac{-q^2}{r_1^3}\cdot((x+\delta)\,\mathbf{i} + y\mathbf{j}) = k_e\frac{-q^2}{r_1^3}\cdot((-1-\delta)\,\mathbf{i}+\mathbf{j}) \\ \mathbf{F}_2 &= k_e\frac{-q^2}{r_2^3}\cdot((1-\delta)\,\mathbf{i}+\mathbf{j}) \\ \mathbf{F}_3 &= k_e\frac{-q^2}{r_2^3}\cdot((1-\delta)\,\mathbf{i}-\mathbf{j}) \\ \mathbf{F}_4 &= k_e\frac{-q^2}{r_1^3}\cdot((-1-\delta)\,\mathbf{i}-\mathbf{j}), \end{aligned} \tag{29}$$

where $r_1 = \sqrt{(x+\delta)^2+y^2} = \sqrt{(1+\delta)^2+1}$ and $r_2 = \sqrt{(1-\delta)^2+1}$.
Taylor expansion of $\frac{1}{r_1^3}$ and $\frac{1}{r_2^3}$ up to first order in δ gives:

$$\begin{aligned} \frac{1}{r_1^3} &= \frac{1}{((1+\delta)^2+1)^{3/2}} \approx \frac{1}{2\sqrt{2}} - \frac{3}{4\sqrt{2}}\,\delta \\ \frac{1}{r_2^3} &= \frac{1}{((1-\delta)^2+1)^{3/2}} \approx \frac{1}{2\sqrt{2}} + \frac{3}{4\sqrt{2}}\,\delta. \end{aligned} \tag{30}$$

As expected by symmetry, when summing up all four forces only a component in the x-direction remains:

$$\mathbf{F} = \sum_{i=1}^{4}\mathbf{F}_i = k_e q^2\cdot 2\cdot\left[\frac{1-\delta}{r_2^3} + \frac{-1-\delta}{r_1^3}\right]\mathbf{i}$$

$$\begin{aligned}
&\overset{eqn.(30)}{\approx} && k_e q^2 \cdot 2 \cdot \Big[(1-\delta)\Big(\frac{1}{2\sqrt{2}} + \frac{3}{4\sqrt{2}}\,\delta\Big) + (-1-\delta)\Big(\frac{1}{2\sqrt{2}} - \frac{3}{4\sqrt{2}}\,\delta\Big)\Big]\,\mathbf{i} \\
&= && k_e q^2 \Big(\frac{3}{\sqrt{2}}\,\delta - \frac{2}{\sqrt{2}}\,\delta\Big)\,\mathbf{i} \\
&= && k_e q^2 \frac{1}{\sqrt{2}}\,\delta\,\mathbf{i}. \qquad (31)
\end{aligned}$$

Thus we have established that a displacement in the positive x-direction will cause a force also in the positive x-direction and the system is unstable. The corresponding spring constant is

$$k = -k_e q^2 \frac{1}{\sqrt{2}} = -\frac{q^2}{4\pi\varepsilon_0\sqrt{2}}\,. \qquad (32)$$

7. The electric field at x of a rod extending from $-a$ to $+a$ which carries a charge Q uniformely distributed on it is

$$E_{rod} = \frac{k_e Q}{(x-a)(x+a)} = \frac{k_e Q}{x^2-a^2}\,, \qquad (33)$$

where the field vector is pointing toward x.
The electric field at x of a charge Q located at $+2a$ on the x-axis is

$$E_{point} = \frac{k_e Q}{(2a-x)^2}\,, \qquad (34)$$

where the field vector is pointing toward x.
The total field for a testcharge inbetween then is

$$E = \frac{k_e Q}{x^2-a^2} - \frac{k_e Q}{(2a-x)^2}. \qquad (35)$$

Setting $E \overset{!}{=} 0$:

$$\begin{aligned}
\Longrightarrow (2a-x)^2 &= x^2 - a^2 \\
\Longrightarrow -4xa + 4a^2 &= -a^2 \\
\Longrightarrow 4xa &= 5a^2 \\
\Longrightarrow x &= \frac{5}{4}a\,. \qquad (36)
\end{aligned}$$

8. The electric field of an infinite line charge is

$$E = \frac{\lambda}{2\pi R\varepsilon_0} \qquad (37)$$

where R is the distance to the electron. The centripetal force is $F = m\frac{v^2}{R}$. Equating the centripetal and coulomb forces yields

$$\begin{aligned} m\frac{v^2}{R} &= \frac{e\lambda}{2\pi R\varepsilon_0} \\ \Longrightarrow \quad v^2 &= \frac{e/m}{2\pi\varepsilon_0}\lambda \\ \Longrightarrow \quad v &= \sqrt{\frac{e/m}{2\pi\varepsilon_0}\lambda} \\ \Longrightarrow \quad v &= 7.97\times 10^7\ \frac{m}{s}\,. \end{aligned} \tag{38}$$

9. First let us calculate the y-component of the electric field.

$$\begin{aligned} E_y &= \mathbf{E}\cdot\mathbf{j} = k_e\int_{y=0}^{\infty}\frac{dq}{r^2}\,(\hat{\mathbf{r}}\cdot\mathbf{j}) \\ &= k_e\int_{y=0}^{\infty}\frac{\lambda dy}{r^2}\left(\frac{-y}{r}\right) \\ &= -\lambda k_e\int_{y=0}^{\infty}\frac{y}{r^3}dy = -\lambda k_e\int_{y=0}^{\infty}\frac{y}{(a^2+y^2)^{3/2}}dy \\ &= \frac{-\lambda k_e}{a}\,. \end{aligned} \tag{39}$$

We could also explicitly calculate the x-component by integrating the appropriate function. But it is much simpler to note that the x-component of the field of an infinite rod is twice the x-component of the field of the semi-infinite rod. Thus

$$E_x = \frac{1}{2}\cdot E_{x,\infty} = \frac{\lambda k_e}{a}\,. \tag{40}$$

10. The problem is symmetric about x, thus there is no field component in the x-direction. The field in the y-direction is:

$$\begin{aligned} E_y &= \mathbf{E} \cdot \mathbf{j} = -k_e \int_{\theta=0}^{\pi} \frac{dq}{r^2} \left(\mathbf{e}_r \cdot \mathbf{j} \right) \\ &= k_e \int_{\theta=0}^{\pi} \frac{\lambda a d\theta}{a^2} \left(-\sin(\theta) \right) \\ &= \frac{k_e \lambda}{a} \int_{\theta=0}^{\pi} -\sin(\theta)\, d\theta \\ &= \frac{k_e \lambda}{a} \left(\cos(\pi) - \cos(0) \right) \\ &= -\frac{2k_e \lambda}{a}\,. \end{aligned} \tag{41}$$