Geometrical Optics R.Shankar

Here are some notes on optics that you may not find anywhere else. Please refer to Fig. 1 that shows a parabolic mirror with

$$y^2 = 4xf. \tag{1}$$

We know it can focus all rays from infinity to the focal point F no matter at what y they hit the mirror. This does not however mean that all rays coming from a source at a *finite* distance will hit the mirror and converge at one point to form the image.

We have seen using ray optics that at least three rays of light from the top of the source, (u, h_1) reach the bottom of the image $(v, -h_2)$: the ray that went horizontally and hit the mirror at a height $y = h_1$ and returned through the focus, one that followed the dotted lines in the figure and got reflected at $y = 0$ and the ray that went through the focal point and hit the mirror at $y = -h_2$ and emerged parallel at height $y = -h_2$.

But how do we know that rays at every other value of y will do the same? Surely we cannot be drawing an infinite number of figures, one for each y, to check this. Instead we will calculate the time for a general height of impact y and see if it equals the time for the dotted path corresponding to $y = 0$, the latter being clearly a path of least time since it obeys $i = r$ upon hitting the vertical segment of the mirror.

Since the velocity is the same along all paths we will compare distances instead of time.

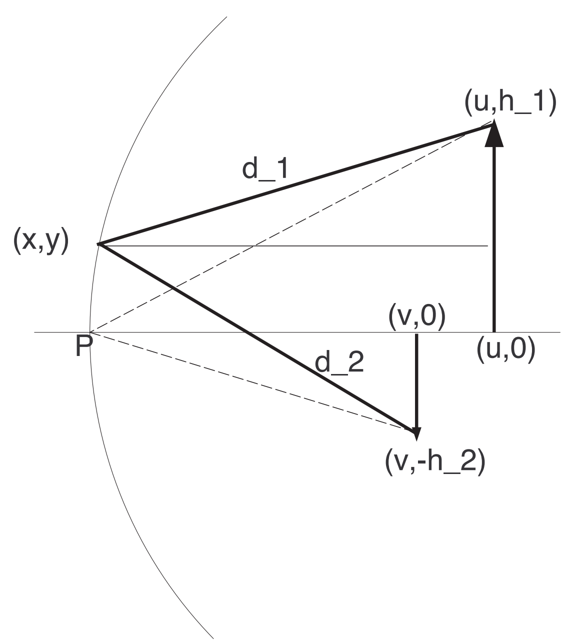

Figure 1: The goal is to equate the distance $d_1 + d_2$ for a general height of impact y to the corresponding distances for $y = 0$, the midpoint P.

It turns out that the time (or distance) actually changes as we vary y away from $y = 0$. However it varies very slowly in the following sense. If we write

$$d(y) = d_1(y) + d_2(y) = d(0) + Ay + by^2 + cy^3 + dy^4... \tag{2}$$

then $A = 0$ and $B = 0$. Thus the function $d(y)$ is very very flat at the bottom, though it eventually starts rising. When we do the calculation, we will find out when that happens.

Let us first recall the approximation due to the binomial theorem

$$(b^2+s^2)^{1/2} = b(1+\frac{s^2}{b^2})^{1/2} = b(1+\frac{s^2}{2b^2} + \text{terms of order } \tfrac{s^4}{b^4} \text{ and higher}) \simeq b + \frac{s^2}{2b} \qquad (3)$$

where b stands for "big" and and s stands for "small". (In class I used u and θ instead.)

In our case u, v, f will stand for big things (b), h_1, h_2, y will stand for small things (s) and $x = y^2/4f$ will stand for a (small)^2 thing (s^2). The product of two small things is like (small)^2. We will only keep terms of order s^2.

Thus we want to consider (see the figure)

$$\begin{aligned}
d(y) - d(0) &= d_1(y) + d_2(y) - d_1(0) - d_2(0) && (4)\\
&= \sqrt{(u-x)^2+(h_1-y)^2} + \sqrt{(v-x)^2+(h_2+y)^2} - \sqrt{u^2+h_1^2} - \sqrt{v^2+h_2^2} && (5)\\
&= \sqrt{(u^2-2ux+x^2+h_1^2+y^2-2yh_1)} + \sqrt{(v^2-2vx+x^2+h_2^2+y^2+2yh_2)} \\
&- \sqrt{u^2+h_1^2} - \sqrt{v^2+h_2^2} && (6)\\
&= u + \frac{-uy^2/2f + h_1^2 + y^2 - 2yh_1}{2u} + v + \frac{-vy^2/2f + h_2^2 + y^2 + 2yh_2}{2v} - u - \frac{h_1^2}{2u} - v - \frac{h_2^2}{2v} \\
&= y\left(\frac{-2h_1}{2u} + \frac{2h_2}{2v}\right) + y^2\left(\frac{1}{2u} + \frac{1}{2v} - \frac{1}{4f} - \frac{1}{4f}\right) && (7)
\end{aligned}$$

where we have dropped the $x^2 \simeq s^4$ term and replaced $x = y^2/4f$ in going from Eq. 5 to Eq. 6, and dropped terms of order s^4/b^4.

Equating the coefficients of y and y^2 to zero we find

$$\frac{h_1}{u} = \frac{h_2}{v} \qquad \frac{1}{u} + \frac{1}{v} = \frac{1}{f}. \qquad (8)$$

It looks like $d(y)$ has only two possible terms in y, linear and quadratic, and we can make both vanish by relating the image parameters v and h_2 in terms of the given object and mirror parameters u and h_1 and f. So we get a sharp image for all y? No! We got this result only to order s^2/b^2. If we expand things out to higher orders we will find additional y -dependence that cannot be eliminated since we have used the freedom to choose v and h_2 for any given u, h_1 and f to kill the y and y^2 terms above.

This means that when the s^4/b^4 terms are not negligible, the rays will no longer meet at one point and the image will be blurred. These terms are of course always there, but the point is that if we look at things to some accuracy, their effect will not be discernable.