Econ 252 - Financial Markets

Spring 2011

Professor Robert Shiller

Problem Set 2 – Solution

Question 1

Denote the portfolio weight of asset A by w (implying that the weight on asset B is 1-w). One obtains

$$E[r_P] = E[w \cdot r_A + (1-w)r_B] = wE[r_A] + (1-w)E[r_B]$$
$$\Leftrightarrow w = \frac{E[r_p] - E[r_B]}{E[r_A] - E[r_B]}$$

for assets A and B as well as

$$E[r_P] = E[w \cdot r_A + (1-w)r_C] = wE[r_A] + (1-w)E[r_C] = wE[r_A] + (1-w)r_C$$
$$\Leftrightarrow w = \frac{E[r_p] - r_C}{E[r_A] - r_C}$$

for assets A and C.

(a) The portfolio weight is

$$w = \frac{E[r_p]-E[r_B]}{E[r_A]-E[r_B]} = \frac{0.07-0.08}{0.05-0.08} = \frac{1}{3}.$$

Using this portfolio weight, it follows that the return variance of the portfolio is

$$Var(r_P) = Var\left(\frac{1}{3}r_A + \frac{2}{3}r_B\right) = \left(\frac{1}{3}\right)^2 Var(r_A) + \left(\frac{2}{3}\right)^2 Var(r_B) + 2\cdot\frac{1}{3}\cdot\frac{2}{3}Cov(r_{A,}r_B)$$
$$= \left(\frac{1}{3}\right)^2 Var(r_A) + \left(\frac{2}{3}\right)^2 Var(r_B) + 2\cdot\frac{1}{3}\cdot\frac{2}{3}Corr(r_{A,}r_B)Std(r_A)Std(r_B)$$
$$= \left(\frac{1}{3}\right)^2 (0.25)^2 + \left(\frac{2}{3}\right)^2 (0.32)^2 + 2\cdot\frac{1}{3}\cdot\frac{2}{3}(-0.3)\cdot 0.25\cdot 0.32$$
$$\approx 0.0418.$$

It follows that the return standard deviation is

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.0418} \approx 0.2045 = 20.45\%.$$

(b) The portfolio weight is

$$w = \frac{E[r_p]-E[r_B]}{E[r_A]-E[r_B]} = \frac{0.03-0.08}{0.05-0.08} = \frac{5}{3}.$$

Using this portfolio weight, it follows that the return variance of the portfolio is

$$Var(r_P) = Var\left(\frac{5}{3}r_A - \frac{2}{3}r_B\right) = \left(\frac{5}{3}\right)^2 Var(r_A) + \left(-\frac{2}{3}\right)^2 Var(r_B) + 2\cdot\frac{5}{3}\cdot\left(-\frac{2}{3}\right)Cov(r_{A,}r_B)$$
$$= \left(\frac{5}{3}\right)^2 Var(r_A) + \left(-\frac{2}{3}\right)^2 Var(r_B) + 2\cdot\frac{5}{3}\cdot\left(-\frac{2}{3}\right)Corr(r_{A,}r_B)Std(r_A)Std(r_B)$$
$$= \left(\frac{5}{3}\right)^2 (0.25)^2 + \left(-\frac{2}{3}\right)^2 (0.32)^2 + 2\cdot\frac{1}{3}\cdot\left(-\frac{2}{3}\right)\cdot(-0.3)\cdot 0.25\cdot 0.32$$
$$\approx 0.2298.$$

It follows that the return standard deviation is

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.2298} \approx 0.4794 = 47.94\%.$$

(c) The portfolio weight is

$$w = \frac{E[r_p] - r_C}{E[r_A] - r_C} = \frac{0.025 - 0.02}{0.05 - 0.02} = \frac{1}{6}.$$

Using this portfolio weight, it follows that the return variance of the portfolio is

$$Var(r_P) = Var\left(\frac{1}{6} r_A + \frac{5}{6} r_C\right) = \left(\frac{1}{6}\right)^2 Var(r_A) + 0 + 0 = \left(\frac{1}{6}\right)^2 (0.25)^2 \approx 0.0017.$$

It follows that the return standard deviation is

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.0017} \approx 0.0412 = 4.12\%.$$

(d) The portfolio weight is

$$w = \frac{E[r_p] - r_C}{E[r_A] - r_C} = \frac{0.10 - 0.02}{0.05 - 0.02} = \frac{8}{3}.$$

Using this portfolio weight, it follows that the return variance of the portfolio is

$$Var(r_P) = Var\left(\frac{8}{3} r_A - \frac{5}{3} r_C\right) = \left(\frac{8}{3}\right)^2 Var(r_A) + 0 + 0 = \left(\frac{8}{3}\right)^2 (0.25)^2 \approx 0.4444.$$

It follows that the return standard deviation is

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.4444} \approx 0.6666 = 66.66\%.$$

Question 2

(a) w=-0.5:

$$E[r_P] = E[-0.5\cdot r_A + 1.5\cdot r_B] = -0.5\cdot E[r_A] + 1.5\cdot E[r_B]$$
$$= -0.5\cdot 0.04 + 1.5\cdot 0.015 = 0.0025 = 0.25\%.$$

$$Var(r_P) = Var(-0.5\cdot r_A + 1.5\cdot r_B)$$
$$= (-0.5)^2\cdot Var(r_A) + (1.5)^2\cdot Var(r_B) + 2\cdot(-0.5)\cdot 1.5\cdot Corr(r_A, r_B)\cdot Std(r_A)\cdot Std(r_B)$$
$$= (-0.5)^2\cdot(0.42)^2 + (1.5)^2\cdot(0.24)^2 + 2\cdot(-0.5)\cdot 1.5\cdot 0.1\cdot 0.42\cdot 0.24 \approx 0.1586.$$

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.1586} \approx 0.3983 = 39.83\%.$$

w=0.3:

$$E[r_P] = E[0.3\cdot r_A + 0.7\cdot r_B] = 0.3\cdot E[r_A] + 0.7\cdot E[r_B]$$
$$= 0.3\cdot 0.04 + 0.7\cdot 0.015 = 0.0225 = 2.25\%.$$

$$Var(r_P) = Var(0.3\cdot r_A + 0.7\cdot r_B)$$
$$= (0.3)^2\cdot Var(r_A) + (0.7)^2\cdot Var(r_B) + 2\cdot 0.3\cdot 0.7\cdot Corr(r_A, r_B)\cdot Std(r_A)\cdot Std(r_B)$$
$$= (0.3)^2\cdot(0.42)^2 + (0.7)^2\cdot(0.24)^2 + 2\cdot 0.3\cdot 0.7\cdot 0.1\cdot 0.42\cdot 0.24 \approx 0.0483.$$

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.0483} \approx 0.2198 = 21.98\%.$$

w=0.8:

$$E[r_P] = E[0.8\cdot r_A + 0.2\cdot r_B] = 0.8\cdot E[r_A] + 0.2\cdot E[r_B]$$
$$= 0.8\cdot 0.04 + 0.2\cdot 0.015 = 0.035 = 3.5\%.$$

$$Var(r_P) = Var(0.8\cdot r_A + 0.2\cdot r_B)$$
$$= (0.8)^2\cdot Var(r_A) + (0.2)^2\cdot Var(r_B) + 2\cdot 0.8\cdot 0.2\cdot Corr(r_A, r_B)\cdot Std(r_A)\cdot Std(r_B)$$
$$= (0.8)^2\cdot(0.42)^2 + (0.2)^2\cdot(0.24)^2 + 2\cdot 0.8\cdot 0.2\cdot 0.1\cdot 0.42\cdot 0.24 \approx 0.1184.$$

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.1184} \approx 0.3441 = 34.41\%.$$

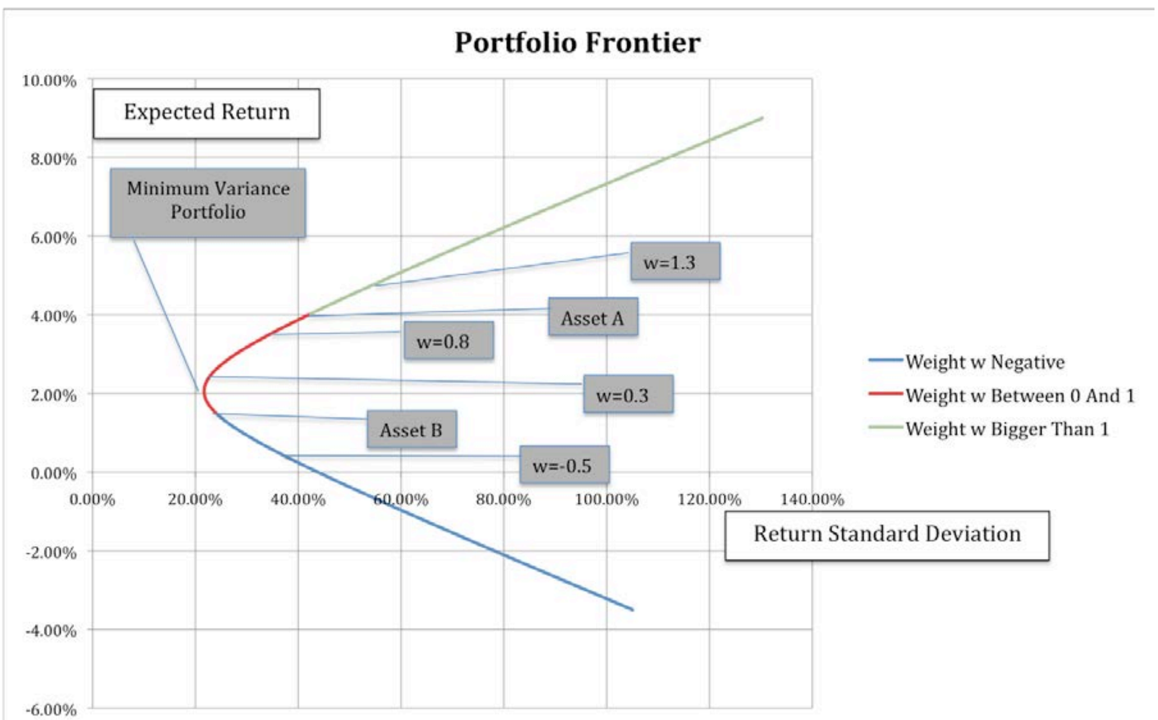
w=1.3:

$$E[r_P] = E[1.3\cdot r_A - 0.3\cdot r_B] = 1.3\cdot E[r_A] - 0.3\cdot E[r_B]$$
$$= 1.3\cdot 0.04 - 0.3\cdot 0.015 = 0.0475 = 4.75\%.$$

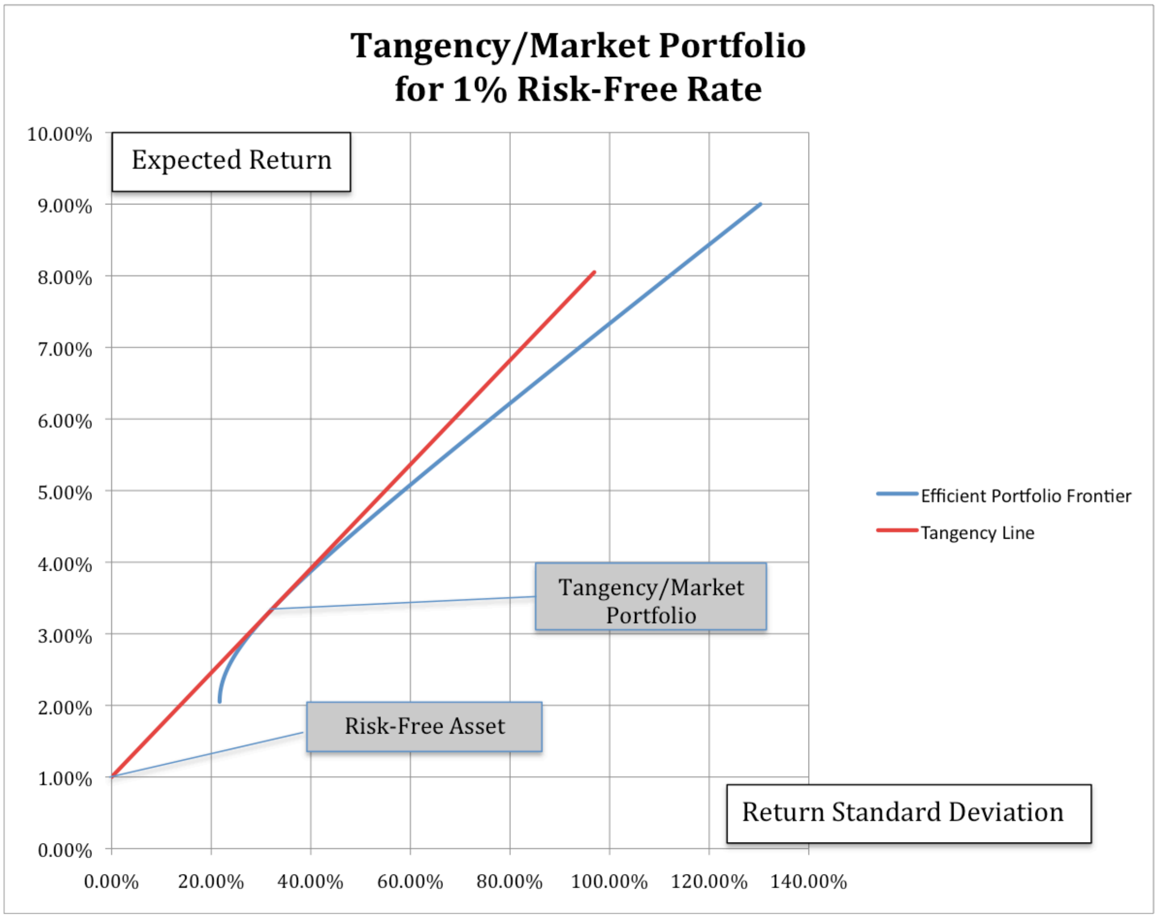
$$Var(r_P) = Var(1.3\cdot r_A - 0.3\cdot r_B)$$
$$= (1.3)^2\cdot Var(r_A) + (-0.3)^2\cdot Var(r_B) + 2\cdot 1.3\cdot(-0.3)\cdot Corr(r_A, r_B)\cdot Std(r_A)\cdot Std(r_B)$$
$$= (1.3)^2\cdot(0.42)^2 + (-0.3)^2\cdot(0.24)^2 + 2\cdot 1.3\cdot(-0.3)\cdot 0.1\cdot 0.42\cdot 0.24 \approx 0.2954.$$

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.2954} \approx 0.5435 = 54.35\%.$$

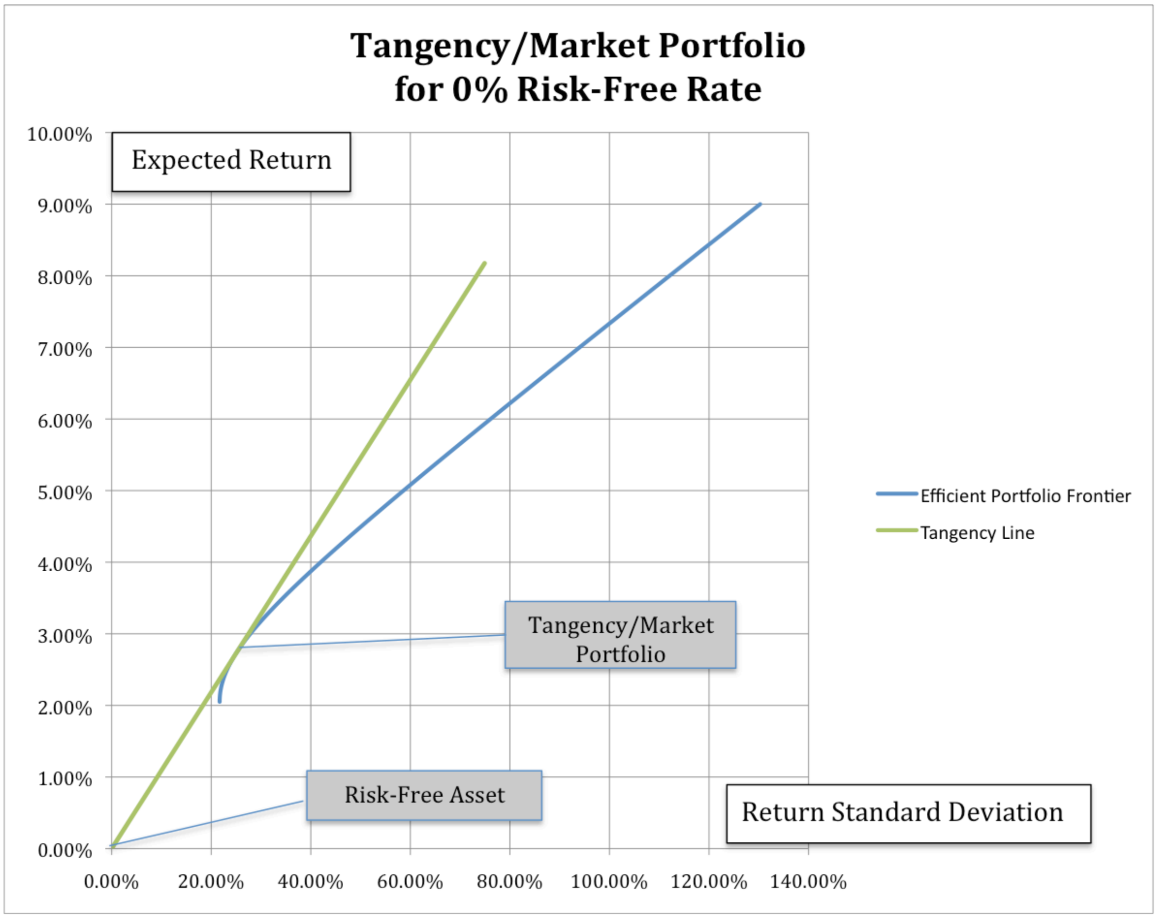
(b)



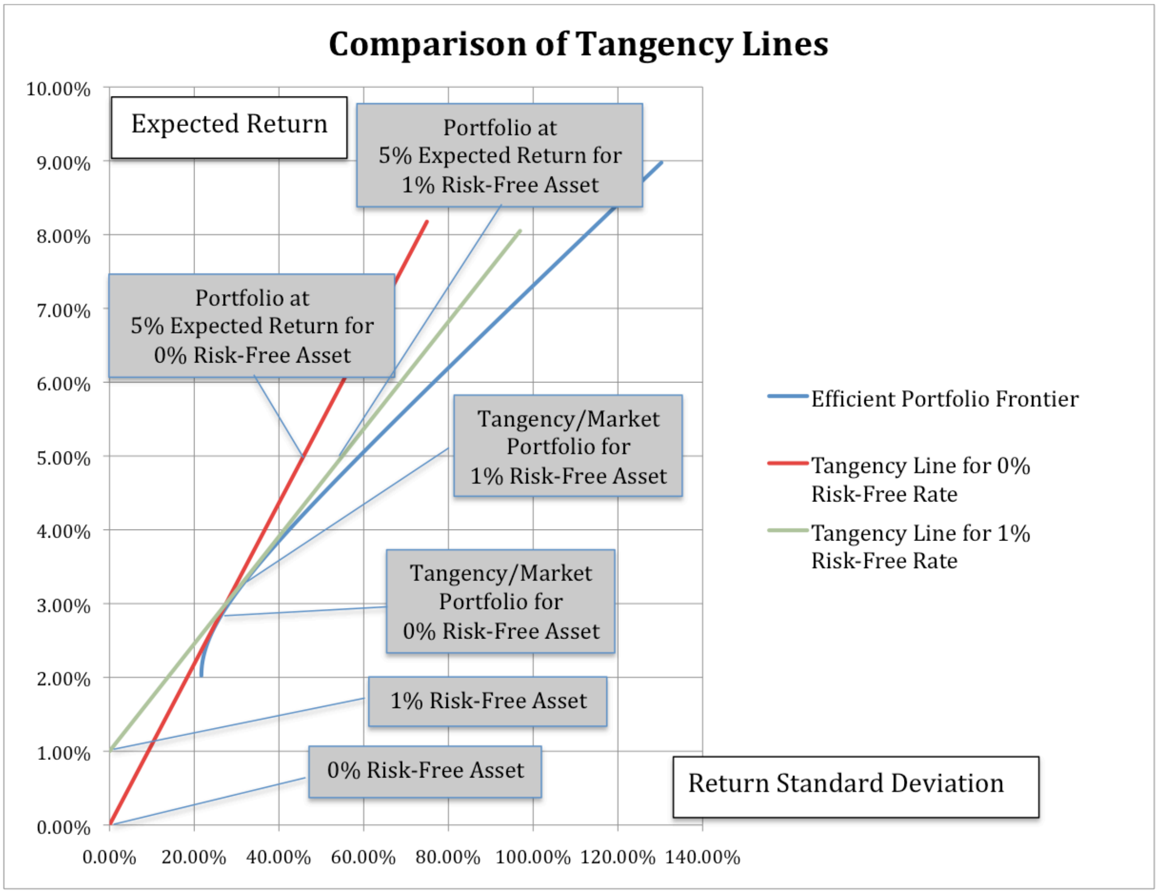
(c)


Tangency/Market Portfolio
for 1% Risk-Free Rate
Expected Return
10.00%
9.00%
8.00%
7.00%
6.00%
5.00%
4.00%
3.00%
2.00%
1.00%
0.00%
0.00%
20.00%
40.00%
60.00%
80.00%
100.00%
120.00%
140.00%
Tangency/Market Portfolio
Risk-Free Asset
Return Standard Deviation
Efficient Portfolio Frontier
Tangency Line

(d)


Tangency/Market Portfolio
for 0% Risk-Free Rate
Expected Return
10.00%
9.00%
8.00%
7.00%
6.00%
5.00%
4.00%
3.00%
2.00%
1.00%
0.00%
0.00%
20.00%
40.00%
60.00%
80.00%
100.00%
120.00%
140.00%
Tangency/Market Portfolio
Risk-Free Asset
Return Standard Deviation
Efficient Portfolio Frontier
Tangency Line

(e)



According to the above graph, the risk-free rate from part (d) (0%) allows for a lower standard deviation on the tangency line at an expected return of 5%. Therefore, the 0% risk-free rate is preferable.

Question 3

(a) It follows from the formula of the Sharpe ratio that

$$SR_M = \frac{E[r_M]-r_f}{Std(r_M)} \Leftrightarrow r_f = E[r_M]-SR_M \cdot Std(r_M).$$

Hence, one obtains

$$r_f = E[r_M]-SR_M \cdot Std(r_M) = 0.12-0.5\cdot 0.2 = 0.02.$$

(b) As all portfolios are located on the Tangency Line, they all have the same Sharpe ratio, in particular their Sharpe ratio equals that of the Market Portfolio.

Therefore, the standard deviation of a representative portfolio P on the Tangency Line satisfies:

$$SR_M = \frac{E[r_P]-r_f}{Std(r_P)} \Leftrightarrow Std(r_P) = \frac{E[r_P]-r_f}{SR_M}.$$

It follows that portfolio 1 has a return standard deviation of

$$Std(r_1) = \frac{E[r_1]-r_f}{SR_M} = \frac{0.063-0.02}{0.5} = 0.086 = 8.6\%.$$

Analogously, portfolio 2 has a return standard deviation of

$$Std(r_2) = \frac{E[r_2]-r_f}{SR_M} = \frac{0.0825-0.02}{0.5} = 0.125 = 12.5\%.$$

Finally, portfolio 3 has a return standard deviation of

$$Std(r_3) = \frac{E[r_3]-r_f}{SR_M} = \frac{0.178-0.02}{0.5} = 0.316 = 31.6\%.$$

(c) Portfolio 1 yields a utility value of

$$u(P1) = 0.063 - 2 \cdot (0.086)^2 \approx 0.048.$$

Portfolio 2 yields a utility value of

$$u(P2) = 0.0825 - 2 \cdot (0.125)^2 \approx 0.0513.$$

Portfolio 3 yields a utility value of

$$u(P3) = 0.178 - 2 \cdot (0.316)^2 \approx -0.022.$$

Therefore, portfolio 2 yields the highest utility and will be chosen by the agent.

(d) Consider a portfolio with weight w on the risk-free asset and weight 1-w on the market portfolio. Its expected return is

$$E[r_P] = E[w\cdot r_f + (1-w)r_M] = wE[r_f] + (1-w)E[r_M]$$
$$= w\cdot 0.02 + (1-w)\cdot 0.12 = 0.12 - 0.1\cdot w.$$

Its return variance is

$$Var(r_P) = Var(w\cdot r_f + (1-w)r_M) = 0 + (1-w)^2 Var(r_M) + 0$$
$$= (1-w)^2\cdot (0.2)^2 = 0.04(1-w)^2.$$

It follows that the agent aims at maximizing

$$0.12 - 0.1\cdot w - 2\cdot 0.04(1-w)^2$$

with respect to w. Setting the first derivative equal to zero, it follows that

$$-0.1 + 2\cdot 0.04\cdot 2(1-w) = 0 \Leftrightarrow 1-w = 0.625 \Leftrightarrow w = 0.375.$$

The SOC states that

$$-2\cdot 0.04\cdot 2 < 0,$$

implying that the above w indeed characterizes a maximum.

w=0.375 implies a portfolio with expected return

$$E[r_P] = 0.12 - 0.1\cdot 0.375 = 0.0825 = 8.25\%$$

and with return variance

$$Var(r_P) = 0.04(1-0.375)^2 = 0.0156 = 1.56\%,$$

resulting in a return standard deviation of 0.125=12.5%, which is exactly the chosen portfolio from part (c).

Question 4

(a) The annual returns of the market index are as follows:

$$2004 \to 2005: \frac{110-100}{100} = 0.1 = 10\%,$$
$$2005 \to 2006: \frac{104.5-110}{110} = -0.05 = -5\%,$$
$$2006 \to 2007: \frac{106.59-104.5}{104.5} = 0.02 = 2\%,$$
$$2007 \to 2008: \frac{106.59-106.59}{106.59} = 0 = 0\%,$$
$$2008 \to 2009: \frac{110.85-106.59}{106.59} \approx 0.04 = 4\%,$$
$$2009 \to 2010: \frac{108.63-110.85}{110.85} = -0.02 = -2\%.$$

Therefore, the expected return of the market index is

$$E[r_M] = \frac{1}{6}(0.1-0.05+0.02+0+0.04-0.02) = 0.015 = 1.5\%.$$

(b) The CAPM implies that

$$E[r_A] - r_f = \beta_A(E[r_M] - r_f) \Leftrightarrow E[r_A] = r_f + \beta_A(E[r_M] - r_f).$$

It follows that the expected return of asset A is

$$E[r_A] = r_f + \beta_A(E[r_M] - r_f) = 0.0075 + 0.8 \cdot (0.015 - 0.0075) = 0.0135 = 1.35\%.$$

(c) The CAPM implies that

$$E[r_B] - r_f = \beta_B(E[r_M] - r_f) \Leftrightarrow E[r_B] = r_f + \beta_B(E[r_M] - r_f).$$

It follows that the expected return of asset B is

$$E[r_B] = r_f + \beta_B(E[r_M] - r_f) = 0.0075 + 3 \cdot (0.015 - 0.0075) = 0.03 = 3\%.$$

(d) The CAPM implies that

$$E[r_C]-r_f=\beta_C(E[r_M]-r_f)\Leftrightarrow \beta_C=\frac{E[r_C]-r_f}{E[r_M]-r_f}.$$

It follows that the beta of asset C is

$$\beta_C=\frac{E[r_C]-r_f}{E[r_M]-r_f}=\frac{0.0075-0.0075}{0.015-0.0075}=0.$$