Econ 252 - Financial Markets

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Problem Set 1 – Solution

Question 1

(a) Denote the winnings from a single lottery ticket by L.

A single lottery ticket pays \$1,000,000 with probability 1/1,000,000, it pays \$10,000 with probability 1/10,000, and it pays \$1 with probability 1/100. Therefore, the expected value of winnings from a single lottery ticket equals

$$E[L] = \frac{1}{1{,}000{,}000} \cdot 1{,}000{,}000 + \frac{1}{10{,}000} \cdot 10{,}000 + \frac{1}{100} \cdot 1 = 2.01.$$

(b) The variance of the winnings from a single lottery ticket equals

$$Var(L) = E[L^2] - E[L]^2$$
$$= \frac{1}{1{,}000{,}000} \cdot (1{,}000{,}000)^2 + \frac{1}{10{,}000} \cdot (10{,}000)^2 + \frac{1}{100} \cdot (1)^2 - (2.01)^2 \approx 1{,}009{,}995.97.$$

(c) The following argument is based on the fact that a potential buyer of a lottery ticket is risk-averse or is risk-neutral.

If the ticket costs \$4, its cost is higher than the expected winnings. In this case, a risk-averse or risk-neutral person would not buy the ticket.

If the ticket costs \$1, its cost is lower than the expected earnings. If someone is risk-neutral or only very weakly risk-averse, this person should buy the ticket. If, however, the person is strongly risk-averse, this person should not buy the ticket.

Question 2

(a) Denote the U.S. bond by US. It pays $100 with probability 1. Therefore,

$$E[US] = 1 \cdot 100 = 100.$$

Denote the NY bond by NY. It pays $100 with probability .3+.15+.05=.5, pays $80 with probability .1+.1+.1=.3, and pays $20 with probability .05+.05+.1=.2. Therefore,

$$E[NY] = .5 \cdot 100 + .3 \cdot 80 + .2 \cdot 20 = 78.$$

Denote the CA bond by CA. It pays $100 with probability .3+.1+.05=.45, pays $80 with probability .15+.1+.05=.3, and pays $20 with probability .05+.1+.1=.25. Therefore,

$$E[CA] = .45 \cdot 100 + .3 \cdot 80 + .25 \cdot 20 = 74.$$

(b) As the U.S. bond pays a fixed amount for sure, its variance equals $0.

The variance of the NY bond equals

$$Var(NY) = E[NY^2] - E[NY]^2 = .5 \cdot (100)^2 + .3 \cdot (80)^2 + .2 \cdot (20)^2 - (78)^2 = 916.$$

The variance of the CA bond equals

$$Var(CA) = E[CA^2] - E[CA]^2 = .45 \cdot (100)^2 + .3 \cdot (80)^2 + .25 \cdot (20)^2 - (74)^2 = 1{,}044.$$

(c) As the variance for the U.S. bond is zero, its standard deviation is also equal to 0. The standard deviation of the NY bond equals

$$Std(NY) = \sqrt{Var(NY)} = \sqrt{916} \approx 30.27.$$

The standard deviation of the CA bond equals

$$Std(CA) = \sqrt{Var(CA)} = \sqrt{1{,}044} \approx 32.31.$$

(d) The covariance of the NY bond and the CA bond equals

$$\begin{aligned} &Cov(NY,CA) = E[NY\cdot CA] - E[NY]E[CA] \\ &= .3\cdot 100\cdot 100 + .15\cdot 100\cdot 80 + .05\cdot 100\cdot 20 \\ &+ .1\cdot 80\cdot 100 + .1\cdot 80\cdot 80 + .1\cdot 80\cdot 20 \\ &+ .05\cdot 20\cdot 100 + .05\cdot 20\cdot 80 + .1\cdot 20\cdot 20 - 78\cdot 74 \\ &= 348. \end{aligned}$$

(e) The correlation of the NY bond and the CA bond equals

$$Corr(NY,CA) = \frac{Cov(NY,CA)}{Std(NY)\cdot Std(CA)} = \frac{348}{30.27\cdot 32.31} \approx 0.3558.$$

(f) The random variable of interest is 1/3 • A + 1/3 • B + 1/3 • C.

The expected value of this random variable is

$$\begin{aligned} &E\left[1/3\cdot US + 1/3\cdot NY + 1/3\cdot CA\right] = 1/3\cdot E[US] + 1/3\cdot E[NY] + 1/3\cdot E[CA] \\ &= 1/3\cdot 100 + 1/3\cdot 78 + 1/3\cdot 74 = 84. \end{aligned}$$

In order to compute the variance of 1/3 • A + 1/3 • B + 1/3 • C, observe that

$$Var(1/3\cdot US + 1/3\cdot NY + 1/3\cdot CA) = Var(1/3\cdot NY + 1/3\cdot CA),$$

as .5 A is a constant. It follows that

$$\begin{aligned} &Var(1/3\cdot US + 1/3\cdot NY + 1/3\cdot CA) = Var(1/3\cdot NY + 1/3\cdot CA) \\ &= Var(1/3\cdot NY) + Var(1/3\cdot CA) + 2\cdot Cov(1/3\cdot NY, 1/3\cdot CA) \\ &= (1/3)^2 Var(NY) + (1/3)^2 Var(C) + 2\cdot 1/3\cdot 1/3\cdot Cov(B,C) \\ &= (1/3)^2\cdot 916 + (.25)^2\cdot 1{,}044 + 2\cdot 1/3\cdot 1/3\cdot 348 \approx 295.11. \end{aligned}$$

Question 3

(a) w=0.75:

$$E[r_P] = E[0.75\cdot r_A + 0.25\cdot r_B] = 0.75\cdot E[r_A] + 0.25\cdot E[r_B]$$
$$= 0.75\cdot 0.1 + 0.25\cdot 0.05 = 0.0875 = 8.75\%.$$

$$Var(r_P) = Var(0.75\cdot r_A + 0.25\cdot r_B)$$
$$= (0.75)^2\cdot Var(r_A) + (0.25)^2\cdot Var(r_B) + 2\cdot 0.75\cdot 0.25\cdot Corr(r_A, r_B)\cdot Std(r_A)\cdot Std(r_B)$$
$$= (0.75)^2\cdot (0.2)^2 + (0.25)^2\cdot (0.15)^2 + 2\cdot 0.75\cdot 0.25\cdot 0.5\cdot 0.2\cdot 0.15 \approx 0.0295.$$

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.0295} \approx 0.1718 = 17.18\%.$$

w=0.5:

$$E[r_P] = E[0.5\cdot r_A + 0.5\cdot r_B] = 0.5\cdot E[r_A] + 0.5\cdot E[r_B]$$
$$= 0.5\cdot 0.1 + 0.5\cdot 0.05 = 0.075 = 7.5\%.$$
$$Var(r_P) = Var(0.5\cdot r_A + 0.5\cdot r_B)$$
$$= (0.5)^2\cdot Var(r_A) + (0.5)^2\cdot Var(r_B) + 2\cdot 0.5\cdot 0.5\cdot Corr(r_A, r_B)\cdot Std(r_A)\cdot Std(r_B)$$
$$= (0.5)^2\cdot (0.2)^2 + (0.5)^2\cdot (0.15)^2 + 2\cdot 0.5\cdot 0.5\cdot 0.5\cdot 0.2\cdot 0.15 \approx 0.0231.$$

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.0231} \approx 0.152 = 15.2\%.$$

w=0.75:

$$E[r_P] = E[0.25\cdot r_A + 0.75\cdot r_B] = 0.25\cdot E[r_A] + 0.75\cdot E[r_B]$$
$$= 0.25\cdot 0.1 + 0.75\cdot 0.05 = 0.0625 = 6.25\%.$$

$$Var(r_P) = Var(0.25\cdot r_A + 0.75\cdot r_B)$$
$$= (0.25)^2\cdot Var(r_A) + (0.75)^2\cdot Var(r_B) + 2\cdot 0.25\cdot 0.75\cdot Corr(r_A, r_B)\cdot Std(r_A)\cdot Std(r_B)$$
$$= (0.25)^2\cdot (0.2)^2 + (0.75)^2\cdot (0.15)^2 + 2\cdot 0.25\cdot 0.75\cdot 0.5\cdot 0.2\cdot 0.15 \approx 0.0208.$$

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.0208} \approx 0.1422 = 14.22\%.$$

In summary,

Weight	Expected Return	Return Standard Deviation
w=0.75	8.75%	17.18%
w=0.50	7.50%	15.20%
w=0.25	6.25%	14.22%

(b) The expected return of each of the three portfolios is not affected by the change in the correlation between assets A and B. It is therefore only necessary to re-compute the return standard deviation for each of the three portfolios.

w=0.75:

$$Var(r_P) = Var(0.75\cdot r_A + 0.25\cdot r_B)$$
$$= (0.75)^2\cdot Var(r_A) + (0.25)^2\cdot Var(r_B) + 2\cdot 0.75\cdot 0.25\cdot Corr(r_A, r_B)\cdot Std(r_A)\cdot Std(r_B)$$
$$= (0.75)^2\cdot (0.2)^2 + (0.25)^2\cdot (0.15)^2 + 2\cdot 0.75\cdot 0.25\cdot (-0.5)\cdot 0.2\cdot 0.15 \approx 0.0183.$$

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.0183} \approx 0.1353 = 13.53\%.$$

w=0.5:

$$Var(r_P) = Var(0.5\cdot r_A + 0.5\cdot r_B)$$
$$= (0.5)^2\cdot Var(r_A) + (0.5)^2\cdot Var(r_B) + 2\cdot 0.5\cdot 0.5\cdot Corr(r_A, r_B)\cdot Std(r_A)\cdot Std(r_B)$$
$$= (0.5)^2\cdot (0.2)^2 + (0.5)^2\cdot (0.15)^2 + 2\cdot 0.5\cdot 0.5\cdot (-0.5)\cdot 0.2\cdot 0.15 \approx 0.0081.$$

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.0081} = 0.09 = 9\%.$$

w=0.25

$$Var(r_P) = Var(0.25\cdot r_A + 0.75\cdot r_B)$$
$$= (0.25)^2\cdot Var(r_A) + (0.75)^2\cdot Var(r_B) + 2\cdot 0.25\cdot 0.75\cdot Corr(r_A, r_B)\cdot Std(r_A)\cdot Std(r_B)$$
$$= (0.25)^2\cdot (0.2)^2 + (0.75)^2\cdot (0.15)^2 + 2\cdot 0.25\cdot 0.75\cdot (-0.5)\cdot 0.2\cdot 0.15 \approx 0.0095.$$

$$Std(r_P) = \sqrt{Var(r_P)} \approx \sqrt{0.0095} \approx 0.0975 = 9.75\%.$$

In summary,

Weight	Expected Return	Return Standard Deviation
w=0.75	8.75%	13.53%
w=0.50	7.50%	9.00%
w=0.25	6.25%	9.75%

Observe the following two properties:

- The two assets that are used to construct each of the above portfolios have standard deviation 20% and 15%. However, there are multiple portfolios whose standard deviation is lower than the standard deviation of the two building blocks. This is a manifestation of the principle of diversification.
- For each of the three portfolio weights, the return standard deviation for -0.5 correlation is strictly lower than the standard deviation for 0.5 correlation. This is a manifestation of the principle that lower correlation provides diversification benefits, which only holds as long as the portfolio weights are between 0 and 1, which they all are in this problem.