Astronomy 160: Frontiers and Controversies in Astrophysics
Homework Set # 3 Solutions

1) **A given object will form a black hole if its radius is less than its Schwarzshild radius. This leads to a very peculiar feature of black holes — the more massive they are, the less dense the material that forms them needs to be. Remember that density is equal to mass divided by volume, or in symbols $\rho = M/((4/3)\pi R^3)$.**

a. **What density would a human being need to be crushed down to in order to become a black hole?**

Assume that a human being weights 100 kg. The equation for finding the density is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

So we can use the definition of the Schwarzschild radius, R_S, for a human being as a black hole.

$$R_S = \frac{2GM}{c^2} = \frac{2 \times 7 \times 10^{-11} \times 100}{9 \times 10^{16}} = 1.5 \times 10^{-25}\ meters$$

So now plug this radius into equation for density:

$$\rho = \frac{100}{4 \times (1.5 \times 10^{-25})^3} = \frac{100}{8 \times 10^{-75}} = 10^{76}\ kg/m^3$$

b. **What density would the Earth need to be crushed down to in order to become a black hole?**

For this part we do the same calculation for the case of the Earth crushed into a black hole.

$$R_S = \frac{2 \times 7 \times 10^{-11} \times 6 \times 10^{24}}{9 \times 10^{16}} = \frac{8 \times 10^{14}}{10^{16}} = 10^{-2}\ meters$$

So then

$$\rho = \frac{6 \times 10^{24}}{4 \times (10^{-2})^3} = \frac{6 \times 10^{24}}{4 \times 10^{-6}} = 1.5 \times 10^{30}\ kg/m^3$$

c. **Suppose you had a huge spherical cosmic ocean of water. How big and how massive would the ocean have to be to form a black hole (without additional compression)?**

$$\rho = \frac{M}{\frac{4}{3}\pi r^3} = \frac{M}{4r^3}$$

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The sphere's radius needs to be at least R_S to become a black hole:

$$R_S = \frac{2GM}{c^2}$$

$$\Downarrow$$

$$M = \frac{R_S c^2}{2G}$$

So the density becomes:

$$\rho = \frac{(\frac{R_S c^2}{2G})}{4R_S^3} = \frac{R_S c^2}{8GR_S^3} = \frac{c^2}{8GR_S^2}$$

$$\Downarrow$$

$$R_S^2 = \frac{c^2}{8G\rho}$$

We know that the density of water is:

$$\rho_{water} = 1000\ kg/m^3$$

so we can now write:

$$R_S^2 = \frac{(3\cdot 10^8)^2}{8\cdot 7\cdot 10^{-11}\cdot 1000} = \frac{9\cdot 10^{16}}{56\cdot 10^{-11}\cdot 10^3} = \frac{90}{56}\cdot 10^{(15+11-3)} = 2\cdot 10^{23}[m^2]$$

$$R_S = \sqrt{2\cdot 10^{23}} = \sqrt{20\cdot 1022} = 4\cdot 10^{11}[m]$$

Now we can calculate the mass of the black hole:

$$M = \frac{R_S c^2}{2G} = \frac{4\cdot 10^{11}\cdot (3\cdot 10^8)^2}{2\cdot 7\cdot 10^{-11}} = \frac{4\cdot 9\cdot 10^{(11+16)}}{14\cdot 10^{-11}} = \frac{36}{14}\cdot 10^{(27+11)}$$

$$M = 2\cdot 10^{38} kg$$

So the radius of a black hole made of uncompressed water would be $4\cdot 10^{11}\ m$ and its mass would be $2\cdot 10^{38}\ kg$.

d. **Derive a general expression relating the mass of a black hole to the density required for a black hole of that mass to form.**

Similarly to the previous section:

$$\rho = \frac{M}{\frac{4}{3}\pi r^3} = \frac{M}{4r^3}$$

$$R_S = \frac{2GM}{c^2}$$

Plugging one into the other:

$$\rho = \frac{M}{4(\frac{2GM}{c^2})^3} = \frac{Mc^6}{4(8G^3M^3)} = \frac{c^6}{32G^3M^2}$$

2) **There is a relativistic expression for the addition of velocities (that is, for the total observed velocity v_{tot} of something that moves at velocity v_1 with respect to another object that itself moves at velocity v_2 with respect to the observer). This expression is**

$$v_{tot} = (v_1 + v_2)/(1 + v_1 v_2/c^2).$$

a. **Show that in the limit where both v_1 and v_2 approach zero,the Newtonian result is recovered.**

In the limit where v_1 and v_2 both approach zero, $v_1 v_2/c2$ becomes very small compared to 1. This means that the "1" becomes the dominant term in the denominator and the summation of velocities becomes:

$$\begin{aligned} v_{\text{tot}} &= \frac{(v_1 + v_2)}{1} \\ &= (v_1 + v_2) \end{aligned}$$

This, of course, is the Newtonian result of the addition of velocities.

b. **Show that if either v_1 or v_2 is equal to c, that v_{tot} is also c. Explain why this latter result shows that the speed of light is the same for all observers.**

If either v_1 or v_2 are equal to c, we are to show that v_{tot} is c. Let's assume $v_2 = c$ and simplify the equation:

$$\begin{aligned} v_{\text{tot}} &= \frac{v_1 + c}{1 + v_1 c/c^2} \\ &= \frac{(v_1 + c)}{1 + v_1/c} \end{aligned}$$

We can then take a factor of c out of the numerator:

$$\begin{aligned} v_{\text{tot}} &= \frac{c(v_1/c + 1)}{1 + v_1/c} \\ &= c\frac{1 + v_1/c}{1 + v_1/c} \\ v_{\text{tot}} &= c \end{aligned}$$

This result shows that the speed of light is the same for all observers because it demonstrates that, no matter what velocity you add to something traveling at the speed of light, the resultant total velocity is still the speed of light. Since v_1 cancels out of the equation, it means that the speed of light doesn't change *no matter what v_1 is equal to.*

c. **Apply the approximation $(1+\epsilon)^n \approx 1+n\epsilon$ to the denominator of the above expression to determine the post-Newtonian correction to the Newtonian result.**

In order to determine the post-Newtonian correction to the Newtonian result, we need to re-write the relativistic summation for velocities:

$$\begin{aligned} v_{\text{tot}} &= \frac{v_1+v_2}{1+v_1v_2/c^2} \\ v_{\text{tot}} &= (v_1+v_2)(1+v_1v_2/c^2)^{-1} \end{aligned}$$

Now, we know that $(1+\epsilon)^n \approx (1+n\epsilon)$. So, we can say:

$$\begin{aligned} v_{\text{tot}} &= (v_1+v_2)(1+v_1v_2/c^2)^{-1} \\ &\approx (v_1+v_2)(1+(-1)(v_1v_2/c^2)) \\ &= (v_1+v_2)(1-v_1v_2/c^2) \\ &= (v_1+v_2)-(v_1+v_2)(v_1v_2/c^2) \\ &= (v_1+v_2)-\frac{v_1^2v_2+v_1v_2^2}{c^2} \end{aligned}$$

The Newtonian addition of velocities can just be expressed as $v_{\text{tot}} = v_1+v_2$. This means that the post-Newtonian correction is just:

$$-\frac{v_1^2v_2+v_1v_2^2}{c^2}$$

3) **Thorne suggests that the questions physicists try to answer are of three kinds: questions about what occurs in the world naturally; questions about what can be accomplished using plausible technologies; and "Sagan questions", which involve infinitely advanced technologies. Pose a question about black holes of each of these kinds (you don't have to answer them!).**

Point breakdown: 1 point for each question (2 out of 3 if you didn't ask questions about black holes, which was in the instructions), 1 point for any reasonable discussion that convinced me you'd given the question some brief thought.

Good question examples - actual student responses:

a) Being that the center of a black hole could possibly be as small as a single point, how do the laws of quantum mechanics affect it?

b) Could we measure relativistic effects near the event horizon of a black hole?

c) Could an infinitely advanced civilization transmit the occurrences inside a black hole to an outside observer? (I am somewhat hesitant about this question, as it implicitly asks 'Could an infinitely advanced civilization "break" the speed of light? and therefore is counter to presently understood laws of physics, but I will leave it, for it seems that the vastness of reality before us has always expanded with the development of technology)