Notes on Images and Potential Problems Physics 201b Spring 2010 Shankar

Here are some notes on solving for V using images. This supplementary material is are for those who want more details. The homework problems and exams will be at the a much simpler level. So read this for fun, not profit.

Before considering images here are some interesting facts about solving for the potential V that will be invoked later on.

1. *The potential V averaged over a sphere that contains no charge equals its value at the center.*

   The proof goes as follows. Consider a nonconducting sphere of radius R with charge Q glued uniformly over its surface. Consider another charge q at a distance r from the center of the sphere. Consider the work needed to assemble this, starting with the sphere at the final location and the point charge q brought in from infinity. Since the field outside the sphere is that of point charge Q, the work done to drag in q is

$$U = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r} \tag{1}$$

   Let us now do the opposite, with the charge q in place and the sphere brought in from infinity. This is the work done to bring each part of the sphere from infinity to its final location. The different points are at different distances r' from the point charge. Thus we may write

$$U = \int_{sphere}\frac{Q}{A}V(r')dA \tag{2}$$

   where A is the area of the sphere. By definition the average of V over the sphere is

$$\bar{V} = \frac{1}{A}\int_{sphere}V(r')dA \tag{3}$$

   Thus

$$U = \int_{sphere}\frac{Q}{A}V(r')dA = Q\bar{V} \tag{4}$$

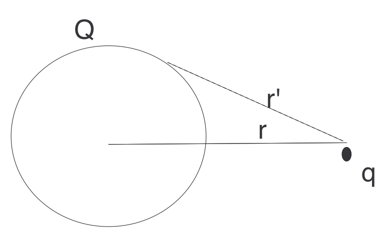

Figure 1: A charge q at a distance of r from the center of a a nonconducting sphere of area A on the surface of which is glued charge Q with uniform density Q/A.

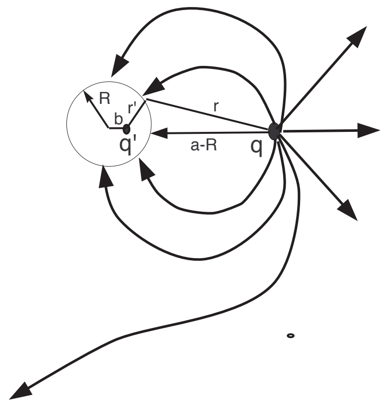

Figure 2: A charge q at a distance of a from the center of a conducting sphere of radius R. (The figure shows the distance from q to the surface of the sphere, namely $a - R$. The image charge q' is in the sphere a distance b from the center.

Comparing Eqns. 1 and 4 we find

$$Q\bar{V} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r} \tag{5}$$

Canceling Q we see that $\bar{V}$ is just the potential of q at the center of the sphere, i.e., $V(0)$.

We can now imagine having any number of charges outside the sphere. Their potentials add and the result applies to the sum since it is true term by term.

2. *In a region free of charges V cannot be a local maximum or minimum.*

   We show this *redictio ad absurdum.* Suppose V has a maximum (or minimum) at some point $\mathbf{r}_0$. Draw a tiny sphere around this point. At all points on the surface of this sphere V is less than (more than) the value at the center. Thus the average over the surface cannot equal the value at the center.

3. *In a region containing any number of conductors at given potentials V_1, V_2.. (including the sphere at infinity at zero potential) there is only one possible $V(\mathbf{r})$.* Suppose there were two potentials $V_A(\mathbf{r})$ and $V_B(\mathbf{r})$ that met these conditions. Their difference $W(\mathbf{r}) = V_A(\mathbf{r}) - V_B(\mathbf{r})$ describes a situation where all conductors are at zero potential. In any other region between conductors $V(\mathbf{r})$ must vanish for if it did not, it would have a maximum or minimum somewhere, which is not allowed. Thus $W \equiv 0$ and $V_A = V_B$.

4. *If the region between conductors has charges at specified locations, then too the potential is unique.*

   Suppose there is some point charge at $\mathbf{r}_0$. Let us surround it by a tiny sphere.