Answers to Midterm Exam

Econ 159a/MGT522a

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- The answers below are more complete than required to get the points. In general, more concise explanations are better.

Question 1. [**15 total points: use blue book 1**]. **Short-Answer Questions**.

State whether each of the following claims is true or false (or can not be determined). For each, explain your answer in (at most) **one short** paragraph**.** Each part is worth **5** points**, of which 4 points are for the explanation.** Explaining an example or a counter-example is sufficient. Absent this, a nice **concise** intuition is sufficient: you do not need to provide a formal proof. Points will be deducted for incorrect explanations.

(a) [**5 points**] "A strictly dominated strategy can never be a best response."

Answer: True. The strategy that strictly dominates it, by definition, yields a strictly higher payoff against all strategies and hence is a better response.

(b) [**5 points**] "In the candidate-voter model, if two people are standing, one to the left of center and one to the right of center, and neither of them is 'too extreme', then it is an equilibrium."

Answer: We accepted true, false or it depends depending on the explanation. This is an equilibrium provided that the players are symmetric around $1/2$*; i.e., equidistant from half. If they are not symmetric, it is not an equilibrium. Notice that players cannot move in the candidate voter model so this is not a possible deviation.*

(c) [**5 points**] "If $(\hat{s}, \hat{s})$ is a Nash equilibrium of a symmetric, two-player game then $\hat{s}$ is evolutionarily stable."

Answer: False. For example consider the game below.

	a	b
a	$1,1$	$0,0$
b	$0,0$	$0,0$

Clearly, (b, b) *is a symmetric NE since* $u(b, b) \geq u(a, b)$*. It is not ES, however, since, we have* $u(b, b) = u(a, b) = 0$*, but* $u(b, a) < u(a, a)$*, a violation of condition* (B)*. Hence a monomorphic population of* b *would be vulnerable to an invasion of mutants that played* a*.*

Question 2. [**30 total points**] **"Party Games"**. Roger has invited Caleb to his party. Roger must choose whether or not to hire a clown. Simultaneously, Caleb must decide whether or not to go the party. Caleb likes Roger but he hates clowns – he even hates other people seeing clowns! Caleb's payoff from going to the party is 4 if there is no clown, but 0 if there is a clown there. Caleb's payoff from not going to the party is 3 if there is no clown at the party, but 1 if there is a clown at the party. Roger likes clowns – he especially likes Caleb's reaction to them — but does not like paying for them. Roger's payoff if Caleb comes to the party is 4 if there is no clown, but $8 - x$ if there is a clown (x is the cost of a clown). Roger's payoff if Caleb does not come to the party is 2 if there is no clown, but $3 - x$ if there is a clown there.

(a) [**6 points**] Write down the payoff matrix of this game.
Answer: *(Throughout, let underlines indicate best-response payoffs.)*

		Caleb	
		go	not go
Roger	Hire	$8-x, \quad 0$	$3-x, \quad \underline{1}$
	Not	$4, \underline{4}$	$2, 3$

(b) [**6 points**] Suppose $x = 0$. Identify any dominated strategies. Explain. Find the Nash equilibrium. What are the equilibrium payoffs?
Answer: *First notice, that no matter what x is, neither of Caleb's strategies are dominated since,*

$$\begin{aligned} u_2(Hire, go) &= 0 < 1 = u_2(Hire, not) \\ \text{and } u_2(Not, go) &= 4 > 3 = u_2(Not, not) \end{aligned}$$

So this will be the case for (c), (d) and (e) below.
Now, when $x = 0$, the payoff matrix becomes:

		Caleb	
		go	not go
Roger	Hire	$\underline{8}, 0$	$\underline{3}, \underline{1}$
	Not	$4, \underline{4}$	$2, 3$

and we see that Hire strictly dominates Not for Roger since,

$$\begin{aligned} u_1(Hire, go) &= 8 > 4 = u_1(Not, go) \\ \text{and } u_1(Hire, not) &= 3 > 2 = u_1(Not, not). \end{aligned}$$

The NE is indicated by the best-response underlining on the matrix, or it can be found as follows. Caleb, knowing that Roger will not play a strictly dominated strategy, should expect Roger to hire a clown. Given Roger is hiring a clown, Caleb's choice is then between choosing go and getting 0 or choosing not and getting 1. Thus the NE is $(Hire, not)$ which yields a payoff of $(3, 1) = \left(\frac{9}{3}, 1\right)$.

(c) [**6 points**] Suppose $x = 2$. Identify any dominated strategies. Explain. Find the Nash equilibrium. What are the equilibrium payoffs?

Answer: *When $x = 2$, the payoff matrix becomes*

		Caleb go	Caleb not go	
Roger	Hire	$\underline{6}, 0$	$1, \underline{1}$	p
	Not	$4, \underline{4}$	$\underline{2}, 3$	$(1-p)$
		q	$(1-q)$	

and we see that neither of Roger's strategies dominates the other since,

$$\begin{aligned} u_1\,(Hire, go) &= 6 > 4 = u_1\,(Not, go) \\ \text{and } u_1\,(Hire, not) &= 1 < 2 = u_1(Not, not). \end{aligned}$$

None of the four pure-strategy profiles has the property that each is strategy a best response to the other, hence there are no pure strategy Nash equilibria. To find the mixed strategy equilibrium, let q denote the probability that Caleb plays go and let p denote the probability that Roger plays $Hire$. In a mixed-strategy Nash equilibrium, q must be such that Roger is indifferent between playing either of her pure strategies, that is, they must both yield the same expected payoff. Similarly, p must be such that Caleb is indifferent between playing either of his pure strategies, that is, they both must yield the same expected payoff.

Hence q must satisfy

$$q \times 6 + (1-q) \times 1 = q \times 4 + (1-q) \times 2 \Rightarrow q = 1/3;$$

and p must satisfy

$$p \times 0 + (1-p) \times 4 = p \times 1 + (1-p) \times 3 \Rightarrow p = 1/2.$$

So mixed strategy Nash equilibrium is $((1/2, 1/2), (1/3, 2/3))$, yielding payoffs of $\left(\frac{8}{3}, 2\right)$.

(d) [6 points] Suppose $x = 3$. Identify any dominated strategies. Explain. Find the Nash equilibrium. What are the equilibrium payoffs?

Answer: *When $x = 3$, the payoff matrix becomes*

		Caleb go	Caleb not go	
Roger	Hire	$\underline{5}, \ 0$	$0, \ \underline{1}$	p
	Not	$4, \underline{4}$	$\underline{2}, 3$	$(1-p)$
		q	$(1-q)$	

and again, we see that neither of Roger's strategies dominates the other since,

$$\begin{aligned} u_1\,(Hire, go) &= 5 > 4 = u_1\,(Not, go) \\ \text{and } u_1\,(Hire, not) &= 0 < 2 = u_1\,(Not, not)\,. \end{aligned}$$

Just as in (c) we see that none of the four pure-strategy profiles has the property that each strategy is a best response to the other, hence there are no pure strategy Nash equilibria. So again to find the mixed strategy equilibrium, let q denote the probability that Caleb plays go and *let* p *denote the probability that Roger plays* $Hire$*..As Caleb's payoff are unchanged from (c)* p *must be the same i.e.* $p = 1/2$*. And* q *must satisfy*

$$q \times 5 + (1-q) \times 0 = q \times 4 + (1-q) \times 2 \Rightarrow q = 2/3.$$

So mixed strategy Nash equilibrium is $((1/2, 1/2), (2/3, 1/3))$ yielding payoffs $\left(\frac{10}{3}, 2\right)$.

(e) [6 points] Suppose $x = 5$. Identify any dominated strategies. Explain. Find the Nash equilibrium. What are the equilibrium payoffs?
Answer: *When* $x = 5$*, the payoff matrix becomes*

		Caleb	
		go	not go
Roger	Hire	$3, 0$	$-2, \underline{1}$
	Not	$\underline{4}, \underline{4}$	$\underline{2}, 3$

Now Roger's strategy Not *strictly dominates* $Hire$ *since,*

$$\begin{aligned} u_1(Hire, go) &= 3 < 4 = u_1(Not, go) \\ \text{and } u_1(Hire, not) &= -2 < 2 = u_1(Not, not). \end{aligned}$$

The NE is indicated by the best-response underlining on the matrix, or it can be found as follows. Caleb, knowing that Roger will not play a strictly dominated strategy, should expect Roger not to hire a clown (i.e. he should anticipate she will choose NOT*). Given Roger is not hiring a clown, Caleb's choice is then between choosing go and getting 4 or choosing not and getting 3. Thus the NE is* (Not, go) *yielding a payoff of* $(4, 4)$*. Notice that compared to the equilibrium from part (a), the higher cost of hiring a clown leads to a outcome that is better for both of them.*

Question 3. [30 total points] "Road Trip". Six Yale students are going on a foreign trip on which they will live close together. Where they are going, there is a disease which spreads easily among people who live close together. The value of the trip to a student who does not get the disease is 6. The value of the trip to a student who gets the disease is 0.

There is a vaccination against the disease. The vaccination costs different amounts for different students (perhaps they have different health plans). Let's call the students $1, 2, 3, 4, 5$ and 6 respectively. The vaccination costs 1 for student 1; it costs 2 for student 2; etc....

If a student gets vaccinated, she will not get the disease. But, if she is not vaccinated then her probability of getting the disease depends on the total number in the group who are not vaccinated. If she is the only person not to get vaccinated then the probability that she gets the disease is $1/6$. If there is one other person who is not vaccinated (i.e., two in all including her) then the probability that she gets the disease is $2/6$. If there are two other people who are not vaccinated (i.e., three including her) then the probability that she gets the disease is $3/6$, etc..

[For example, suppose only students 2 and 4 get vaccinated. Then 2's expected payoff is $6 - [2]$ where the $[2]$ is the cost of the vaccination. Student 4's expected payoff in this case is $6 - [4]$. Student 5's expected payoff in this case (recall she did not get vaccinated) is $\left(\frac{2}{6}\right) 6 + \left(\frac{4}{6}\right) 0 = 2$ where the fraction $\left(\frac{4}{6}\right)$ is the probability that she gets the disease.]

To make this into a game, suppose that each student aims to maximize her expected payoff. The students decide, individually and simultaneously, whether or not to get a vaccination.

(a) [8 points] Explain concisely whether or not it is a Nash equilibrium for students 1,2,3 and 4 to get vaccinated and students 5 and 6 not to get vaccinated.
Answer: *To show that it is not a NE, we only need to find one player who has a strictly profitable deviation. Consider student 4. Her cost of getting vaccinated is* 4, *hence her payoff from getting vaccinated is* $6 - 4 = 2$. *If she instead decided not to get vaccinated as there would be three people (including herself) who were not vaccinated, her payoff would be*

$$\left(\frac{3}{6}\right) \times 6 + \left(\frac{3}{6}\right) \times 0 = 3 > 2.$$

So student 4 *is not playing best response, which means this particular strategy profile is not a Nash equilibrium.*

(b) [8 points] Explain concisely whether or not it is a Nash equilibrium for students 1,2 and 3 to get vaccinated and students 4,5, and 6 not to get vaccinated.
Answer: *We will show that no student has a profitable deviation. First consider those students who are getting vaccinated; in particular, start with student* 3. *Her cost of getting vaccinated is* 3, *hence her payoff from getting vaccinated is* $6 - 3 = 3$. *If she instead decided not to get vaccinated, as there would be four people (including herself) who were not vaccinated, her payoff would be*

$$\left(\frac{2}{6}\right) \times 6 + \left(\frac{4}{6}\right) \times 0 = 2 < 3.$$

So student 3 *is playing a best response. Since the vaccination costs students* 1 *and* 2 *less than it does student* 3, *both students* 1 *and* 2 *have higher payoffs than student* 3 *from getting vaccinated.*

Each of these two students would have the same payoff as would student 3 if they deviated to not getting vaccinated. So they are also playing best responses by choosing to get vaccinated.

Next consider the group of students who are choosing not to get vaccinated. Since there are three students who are not getting vaccinated each has an expected payoff of

$$\left(\frac{3}{6}\right) \times 6 + \left(\frac{3}{6}\right) \times 0 = 3.$$

Student 4's payoff if she chose to get vaccinated would be $6 - 4 = 2$. So she is playing a best response. As the other two students have higher costs of getting vaccinated than student 4, the payoff they would get by choosing to get vaccinated would be even lower. So they are also both playing a best response. Hence all six students are playing best responses, and this is a Nash equilibrium.

(c) [6 points] Which players in this game have strictly or weakly dominated strategies? Explain your answers carefully including whether any domination is strict or weak.

Answer: *First, consider student 1. She has the lowest cost of getting vaccinated. Her payoff from getting vaccinated is $6-1 = 5$. The highest payoff she could get from not getting vaccinated is when the other five students all get vaccinated in which case her expected payoff from choosing not to get vaccinated would be*

$$\left(\frac{5}{6}\right) \times 6 + \left(\frac{1}{6}\right) \times 0 = 5.$$

Thus for student 1 getting vaccinated weakly dominates not getting vaccinated. For student 2 (respectively, 3, 4, 5 and 6), as she faces a higher cost of vaccination, getting vaccinated yields a lower payoff than not getting vaccinated when her five companions all choose to get vaccinated. Hence for all students except student 1, getting vaccinated does not dominate not getting vaccinated.

Now consider student 6. She has the highest cost of getting vaccinated. Her payoff from getting vaccinated is $6 - 6 = 0$. The lowest payoff she could get from not getting vaccinated is when none of her five companions get vaccinated in which case student 6's expected payoff from choosing not to get vaccinated would be

$$\left(\frac{0}{6}\right) \times 6 + \left(\frac{6}{6}\right) \times 0 = 0.$$

Thus for student 6 not getting vaccinated weakly dominates getting vaccinated. For student 5 (respectively, 4, 3, 2 and 1), not getting vaccinated yields a lower payoff than getting vaccinated when her five companions all choose not to get vaccinated. Hence for all students except student 6, not getting vaccinated does not dominate getting vaccinated.

(d) [4 points] If we delete all strictly and weakly dominated strategies from all players, which players in the game now have (iteratively) strictly or weakly dominated strategies? Explain carefully.

Answer: *Suppose students 1 and 6 do not play their weakly dominated strategies, that is, suppose student 1 is choosing to get vaccinated and student 6 is choosing not to get vaccinated.*

Consider student 2. Her payoff from getting vaccinated is $6 - 2 = 4$. In this reduced game (where 6 is choosing not to get vaccinated) the highest payoff she could get from not getting *vaccinated is when the remaining four students all get vaccinated. In this case her expected payoff from choosing not to get vaccinated would be*

$$\left(\frac{4}{6}\right) \times 6 + \left(\frac{2}{6}\right) \times 0 = 4.$$

Thus for student 2*, getting vaccinated weakly dominates not getting vaccinated. By a similar argument to the one given in (c) above, getting vaccinated does not weakly dominate not getting vaccinated for students* 3*,* 4 *and* 5*.*

Now consider student 5*. Her payoff from getting vaccinated is* $6 - 5 = 1$*. In this reduced game, since student* 1 *is choosing to get vaccinated, the lowest payoff student* 5 *could get from not getting vaccinated is when none of students* 2*,* 3*, and* 4 *(and of course* 6*) choose to get vaccinated. In this case student* 5*'s expected payoff from choosing not to get vaccinated would be*

$$\left(\frac{1}{6}\right) \times 6 + \left(\frac{5}{6}\right) \times 0 = 1.$$

Thus for student 5 *not getting vaccinated weakly dominates getting vaccinated. By a similar argument to the one given in (c) above, getting vaccinated does not weakly dominate not getting vaccinated for students* 4*,* 3 *and* 2*.*

Some students went further (which turns out to be helpful in answering the last part). Doing a second round of elimination of weakly dominated strategies just leaves students 3 *and* 4*. Let* V *denote the strategy "get vaccinated" and let* D *denote the strategy "don't get vaccinated." Once we fix player* 1*'s and player* 2*'s strategies as getting vaccinated, and fix player* 5*'s and player* 6*'s strategy as not getting vaccinated then all we are left with is players* 3 *and* 4*. We can write down player* 3*'s and player* 4*'s payoffs (taking as given the actions of players 1,2,5 and 6). This gives us a reduced game matrix as follows (where* V *means vaccinate and* D *means don't):*

		Student 4	
		V	D
Student 3	V	$\underline{3}, 2$	$\underline{3}, \underline{3}$
	D	$\underline{3}, \underline{2}$	$2, \underline{2}$

From this matrix we see that for student 3*,* V *weakly dominates* D *and for student* 4*,* D *weakly dominates* V*. Hence a third round of elimination of weakly dominated strategies yields the equilibrium strategy profile, considered in part (b).*

(e) [4 points] [Harder] Find *all* the (possibly mixed) NE in this game. Explain.

Answer: *From parts (b) and (d) we have seen that it is a Nash equilibrium for students* 1*,* 2 *and* 3 *to get vaccinated and students* 4*,* 5 *and* 6 *not to get vaccinated. Notice also that from the payoff matrix of the reduced game we used in part (d), we see that for student* 3*,* D *is a best response to student* 4 *playing* V*, and for student* 4*,* V *is a best response to student* 3 *playing* D*. Hence it is also a Nash equilibrium for* 1*,* 2 *and* **4** *to get vaccinated and students* **3***,* 5 *and* 6 *not to get vaccinated.*

Could there also be other equilibria? Recall that player 1 *had a weakly dominant strategy to get vaccinated. Hence for player* 1 *to put any weight on not getting vaccinated as part of an equilibrium (in which he must be playing a best response), it would have to be the case that* *all the other players (including player 6) were getting vaccinated. Similarly, player 6 had a weakly dominant strategy not to get vaccinated. Hence for player 1 to put any weight on getting vaccinated as part of an equilibrium (in which he must be playing a best response), it would have to be the case that all the other players (including player 1) were not getting vaccinated. But these two things are mutually inconsistent. Hence, in any equilibrium player* 1 *and* 6 *must play their weakly dominant strategies.*

Once we know this, a similar argument rules there being an equilibrium in which player 2 *puts any weight on not getting vaccinated and rules out an equilibrium in which player* 5 *puts any weight on getting vaccinated. That is, any equilibrium must involve player* 2 *getting vaccinated and player* 5 *not getting vaccinated. Thus the only strategies left to determine in equilibrium are those of players* 3 *and* 4*, and we have already found all the pure-strategy equilibria involving those choices.*

Could there be other mixed-strategy equilibria involving mixing by players 3 and 4. To see that no other such equilibrium exists, consider again the matrix that represents the choices and payoffs of players 3 *and* 4 *taking as given the undominated choices of players* $1, 2, 5$ *and* 6*. Let* p *(respectively,* q*) denote the probability that student* 3 *(respectively, student* 4*) chooses* V*.*

		Student 4		
		V	D	
Student 3	V	$3,2$	$3,3$	p
	D	$3,2$	$2,2$	$(1-p)$
		q	$(1-q)$	

In a mixed-strategy Nash equilibrium, q *must be such that Student* 3 *is indifferent between playing either of her pure strategies, that is, they must both yield the same expected payoff. Similarly,* p *must be such that Student* 4 *is indifferent between playing either of her pure strategies, that is, they both must yield the same expected payoff.*

Hence q *must satisfy*

$$q \times 3 + (1-q) \times 3 = q \times 3 + (1-q) \times 2 \Rightarrow q = 1;$$

and p *must satisfy*

$$p \times 2 + (1-p) \times 2 = p \times 3 + (1-p) \times 2 \Rightarrow p = 0.$$

That is, the 'mixed-strategy' Nash equilibrium is $((0,1),(1,0))$*. But that just corresponds to the pure strategy equilibrium* (D, V) *that we already found.* ■