Angular momentum in d=3: some notes. R.Shankar Fall 2006

I thought a few words on angular momentum in $d = 3$ would be helpful.

Before starting convince yourself that the magnitude of $\mathbf{A} \times \mathbf{B}$ is the area of a parallelogram bounded by the two vectors. (Figure(1)).

First, like torque or moment of inertia I, $\mathbf{L}$ is defined only with respect to some origin.

For some origin, a particle located at $\mathbf{r}$ and moving with momentum $\mathbf{p}$ we define

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{1}$$

The vector $\mathbf{L}$ is perpendicular to the plane defined by $\mathbf{r}$ and $\mathbf{p}$. For example if they both lie in the $x - y$ plane, $\mathbf{L}$ will point along the z-axis. To decide if it is up or down the z-axis you need to use the right hand rule. For example if $\mathbf{r} = \; r\,\mathbf{i}$ is along x and $\mathbf{p} = \; p\,\mathbf{j}$ is along y, the $\mathbf{L}$ is up the z-axis (since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$) and has magnitude rp.

Note that a particle does not have to orbit this origin to have angular momentum around the origin. Look at Figure (2).

The dotted line is the linear trajectory of a particle of fixed $\mathbf{p}$. What is $\mathbf{L}$? First consider the time when it is at A when the line joining it to the origin is perpendicular to its momentum. If this distance is r_{A} then $\mathbf{L} = r_{\mathrm{A}}\; p\; \mathbf{k}$. (To get the direction right, draw the vectors $\mathbf{r}$ and $\mathbf{p}$ emanating from the same point and turn the screw driver from $\mathbf{r}$ to $\mathbf{p}$.) Note that the magnitude of L is twice the are of the triangle of base p and height r_{A}.

At a later time when it is at B, $\mathbf{L} = r_{\mathrm{B}} p\; \sin\theta\; \mathbf{k}$. But geometrically $r_{\mathrm{B}} \sin\theta = r_{\mathrm{A}}$ so that $\mathbf{L}$ has the same magnitude. Since it is always along $\mathbf{k}$, $\mathbf{L}$ itself is the same, or conserved for a free particle with no forces on it. Geometrically L is twice the area of a triangle with base p and height $r_{\mathrm{B}} \sin\theta = r_{\mathrm{A}}$ (area bounded by dotted line, the position vector $\mathbf{r}$ and momentum $\mathbf{p}$.) As time goes by the base and height remain same but the other two sides grow indefinitely.

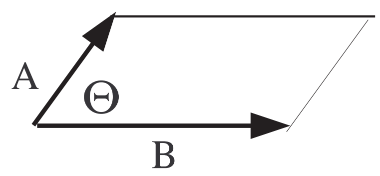

FIG. 1. The cross product of two vectors has a magnitude equal tot he area of the parallelogram defined by them.

A particle whose $\mathbf{p}$ is headed straight for the origin has no angular momentum *with respect to the origin* just like a force aimed at the point of torque computation causes no torque *about that point.*

Now for some amusing uses of these new ideas.

Consider a planet moving under gravity with the sun at the origin. Then

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = 0 \tag{2}$$

since $\mathbf{F}$ and $\mathbf{r}$ are parallel. Thus angular momentum is conserved if the force is central, i.e., pointed towards the origin at all times. Let us draw this $\mathbf{L}$ as a vector in some direction. It never changes as the planet moves. This means the $\mathbf{r}$ and $\mathbf{p}$ vectors always lie in the plane perpendicular to $\mathbf{L}$. That the orbit in a central force lies in a plane is not obvious and follows from the fact $\mathbf{L}$ has fixed direction.

But $\mathbf{L}$ also has fixed magnitude and let us what that implies. Consider Figure (3) which shows an orbit chosen to lie in the plane of the paper.

Let the planet move from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$ in a time dt. The area dA swept out is roughly that of a triangle of height r and base $dr \sin\theta$ (upon ignoring the tiny piece quadratic in small quantities, proportional to dr and $d\theta$.) That is

$$\Delta A = \frac{1}{2} r |d\mathbf{r}| \sin\theta \tag{3}$$

so that upon dividing by dt and taking limit

$$\frac{dA}{dt} = \frac{1}{2} r v \sin\theta = \frac{L}{2m} = \text{ constant in time.} \tag{4}$$

Thus the planar orbit and equal areas in equal times are both due the conservation of $\mathbf{L}$ which in turn is due to the central nature of the force.

However the fact that the orbit closes (into an ellipse) is a special property of the $1/r^2$ force. For example it will not work if it is a $1/r^{1.99}$ force.

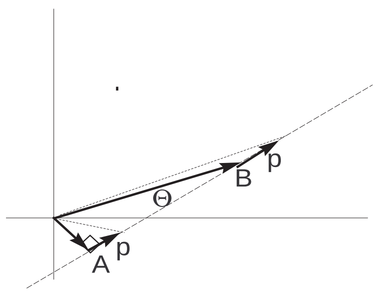

FIG. 2. The angular momentum of the particle is the product of its momentum and perpendicular distance from the origin. This does not change with time if it is free.

Some optional topics for the diehards.

(i) Show the torque due to the pull of gravity of an extended body is obtained by placing all its mass at the CM. (I showed this for the simple case where the body was a straight line and gravity was acting perpendicular to it. You should use the cross product)

(ii) Show that

$$\frac{d\mathbf{L}_T}{dt} = \boldsymbol{\tau}_{ext} \tag{5}$$

where $\mathbf{L}_T$ is the total angular momentum of a collection of particles and $\boldsymbol{\tau}_{ext}$ is the net external torque. To do this you need to break down the force on each mass as external plus internal and consider any pair of particles i and j and show the internal torques cancel. Note that $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ is not enough since the forces act at different locations and could produce different torques. However their sum is clearly proportional to the separation vector. If $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ lie along $\mathbf{r}_i - \mathbf{r}_j$, i.e., the mutual force is central, the cross product will vanish and the internal torques will cancel.

At a fundamental level all forces seem to be central: if two bodies in the universe are exerting forces on each other the only direction for this force that stands out is the line of separation, there is no reason for nature to choose any other direction. Thus angular momentum conservation is due to the fact that free space by itself does not choose one direction over the other.

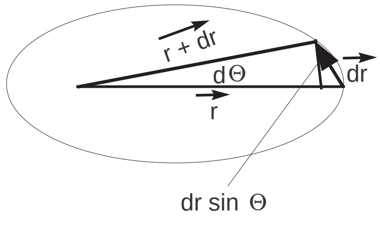

FIG. 3. The area swept out in a small time dt is that of a triangle of height r and base $dr \sin\theta$. But since $dr = vdt$ it follows $dA = vr\sin\theta\ dt$ is essentially $L\ dt$ and hence dA/dt is a constant.