

## Correction to Lecture 19 on the energy in a vibrating string.

I thank Barry Bradlyn for pointing out the error.

Let the displacement of the string be

$$\psi(x, t) = A \cos(kx - \omega t). \quad (1)$$

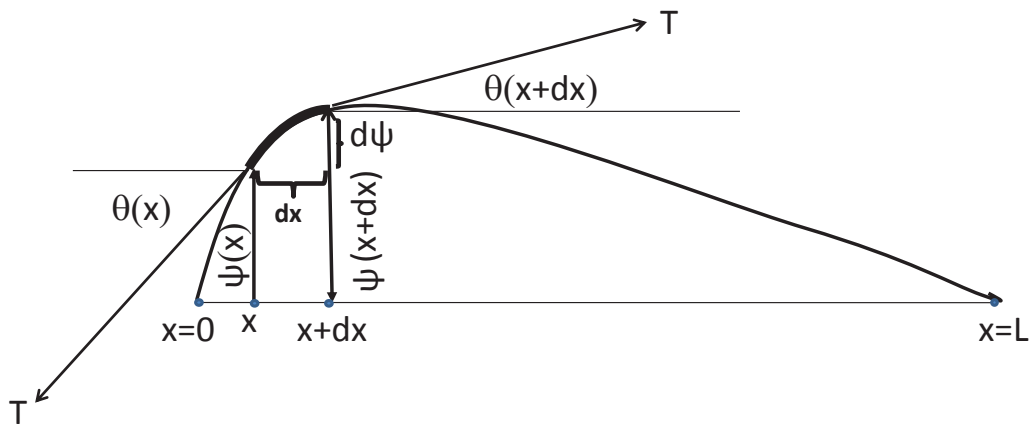


Figure 1: The string under tension  $T$ , has mass  $\mu$  per unit length and is fixed at  $x = 0$  and  $x = L$ . The highlighted segment has a width  $dx$ , with the same tension  $T$  pulling the two ends but at slightly different angles. The displacement  $\psi$  and angles  $\theta$  are exaggerated for clarity. The derivation is valid only when all these are very small.

Now, if it's an infinitely long string, the energy in it is infinite, so you define the energy per unit length. Take a portion of the string, associated with a segment of width  $dx$ , and ask, "How much energy does it have?"

The energy has a kinetic part and a potential part. The kinetic part is simple:

$$dK = \frac{1}{2} \mu dx \left[ \frac{\partial \psi}{\partial t} \right]^2. \quad (2)$$

This is just the mass of the segment times the square of the velocity. For the solution in Eqn. 1 this becomes

$$dK = \frac{1}{2} \mu dx A^2 \omega^2 \sin^2(kx - \omega t). \quad (3)$$

The potential energy  $dU$  has a more subtle origin. The string is under tension  $T$ . When the string is displaced from equilibrium, a segment associated with the interval  $dx$  has a

length (see Figure 1)

$$dl = \sqrt{dx^2 + (d\psi)^2} = dx \sqrt{1 + \left[ \frac{\partial\psi}{\partial x} \right]^2} \simeq dx \left( 1 + \frac{1}{2} \left[ \frac{\partial\psi}{\partial x} \right]^2 \right). \quad (4)$$

Thus it has expanded by an amount

$$\delta l = dl - dx = \frac{1}{2} \left[ \frac{\partial\psi}{\partial x} \right]^2 dx \quad (5)$$

against a tension  $T$ . So the work pumped in and stored as potential energy is

$$dU = T\delta l = \frac{1}{2}T \left[ \frac{\partial\psi}{\partial x} \right]^2 dx. \quad (6)$$

For the solution in Eqn. 1 this becomes

$$dU = \frac{1}{2}Tk^2 A^2 \sin^2(kx - \omega t) \cdot dx = \frac{1}{2}\mu A^2 \omega^2 \sin^2(kx - \omega t) \cdot dx = dK \quad (7)$$

upon using

$$Tk^2 = T \frac{\omega^2}{v^2} = T\omega^2 \frac{\mu}{T} = \mu\omega^2. \quad (8)$$

Notice that the kinetic and potential energies are equal at every value of  $x$  and  $t$ . In particular when the string bit is maximally displaced, it has no velocity and hence no  $dK$ . It also has no  $dU$  because the slope is zero and so is the stretch  $\delta l$ . When the displacement is zero, it has the greatest velocity and slope and hence the greatest  $dK$  and  $dU$ . This is unlike the harmonic oscillator, where due to the conservation of energy,  $K$  and  $U$  add up to a constant, and when one is large the other is small. This is not a problem for the string bit because it is not an isolated system, and energy is pumped into (or out of) it by its neighbors.

The total energy is

$$dE = dK + dU = 2dK = \mu dx A^2 \omega^2 \sin^2(kx - \omega t), \quad (9)$$

The energy per unit length or *energy density*  $u$  is then

$$u(x, t) = \mu A^2 \omega^2 \sin^2(kx - \omega t) \quad (10)$$

Of interest is the *average energy density*  $\bar{u}$ , where the average is over a full cycle in space (a wavelength) or a full cycle in time (a full period  $T$ ). The average of  $\sin^2$  is  $\frac{1}{2}$  in either case and we have the final result

$$\bar{u} = \frac{1}{2}\mu A^2 \omega^2. \quad (11)$$