Question 1

(a) Denote the winnings from a single lottery ticket by $L$.

A single lottery ticket pays $1,000,000 with probability $1/1,000,000$, it pays $10,000$ with probability $1/10,000$, and it pays $1$ with probability $1/100$. Therefore, the expected value of winnings from a single lottery ticket equals

$$E[L] = \frac{1}{1,000,000} \cdot 1,000,000 + \frac{1}{10,000} \cdot 10,000 + \frac{1}{100} \cdot 1 = 2.01.$$ 

(b) The variance of the winnings from a single lottery ticket equals

$$\text{Var}(L) = E[L^2] - (E[L])^2$$

$$= \frac{1}{1,000,000} \cdot (1,000,000)^2 + \frac{1}{10,000} \cdot (10,000)^2 + \frac{1}{100} \cdot (1)^2 - (2.01)^2 \approx 1,009,995.97.$$ 

(c) The following argument is based on the fact that a potential buyer of a lottery ticket is risk-averse or is risk-neutral.

If the ticket costs $4$, its cost is higher than the expected winnings. In this case, a risk-averse or risk-neutral person would not buy the ticket.

If the ticket costs $1$, its cost is lower than the expected earnings. If someone is risk-neutral or only very weakly risk-averse, this person should buy the ticket. If, however, the person is strongly risk-averse, this person should not buy the ticket.
Question 2

(a) Denote the U.S. bond by US. It pays $100 with probability 1. Therefore,

\[ E[US] = 1 \cdot 100 = 100. \]

Denote the NY bond by NY. It pays $100 with probability \(.3 + .15 + .05 = .5\), pays $80 with probability \(.1 + .1 + .1 = .3\), and pays $20 with probability \(.05 + .05 + .1 = .2\). Therefore,

\[ E[NY] = .5 \cdot 100 + .3 \cdot 80 + .2 \cdot 20 = 78. \]

Denote the CA bond by CA. It pays $100 with probability \(.3 + .1 + .05 = .45\), pays $80 with probability \(.15 + .1 + .05 = .3\), and pays $20 with probability \(.05 + .1 + .1 = .25\). Therefore,

\[ E[CA] = .45 \cdot 100 + .3 \cdot 80 + .25 \cdot 20 = 74. \]

(b) As the U.S. bond pays a fixed amount for sure, its variance equals $0.

The variance of the NY bond equals

\[ Var(NY) = E[NY^2] - E[NY]^2 = .5 \cdot (100)^2 + .3 \cdot (80)^2 + .2 \cdot (20)^2 - (78)^2 = 916. \]

The variance of the CA bond equals

\[ Var(CA) = E[CA^2] - E[CA]^2 = .45 \cdot (100)^2 + .3 \cdot (80)^2 + .25 \cdot (20)^2 - (74)^2 = 1,044. \]

(c) As the variance for the U.S. bond is zero, its standard deviation is also equal to 0.

The standard deviation of the NY bond equals

\[ Std(NY) = \sqrt{Var(NY)} = \sqrt{916} \approx 30.27. \]

The standard deviation of the CA bond equals

\[ Std(CA) = \sqrt{Var(CA)} = \sqrt{1,044} \approx 32.31. \]
(d) The covariance of the NY bond and the CA bond equals
\[ \text{Cov}(NY, CA) = E[NY \cdot CA] - E[NY]E[CA] \]
\[ = .3 \cdot 100 \cdot 100 + .15 \cdot 100 \cdot 80 + .05 \cdot 100 \cdot 20 \]
\[ + .1 \cdot 80 \cdot 100 + .1 \cdot 80 \cdot 80 + .1 \cdot 80 \cdot 20 \]
\[ + .05 \cdot 20 \cdot 100 + .05 \cdot 20 \cdot 80 + .1 \cdot 20 \cdot 20 - 78 \cdot 74 \]
\[ = 348. \]

(e) The correlation of the NY bond and the CA bond equals
\[ \text{Corr}(NY, CA) = \frac{\text{Cov}(NY, CA)}{\text{Std}(NY) \cdot \text{Std}(CA)} = \frac{348}{30.27 \cdot 32.31} \approx 0.3558. \]

(f) The random variable of interest is \( \frac{1}{3} \cdot A + \frac{1}{3} \cdot B + \frac{1}{3} \cdot C. \)
The expected value of this random variable is
\[ E\left[ \frac{1}{3} \cdot US + \frac{1}{3} \cdot NY + \frac{1}{3} \cdot CA \right] = \frac{1}{3} \cdot E[US] + \frac{1}{3} \cdot E[NY] + \frac{1}{3} \cdot E[CA] \]
\[ = \frac{1}{3} \cdot 100 + \frac{1}{3} \cdot 78 + \frac{1}{3} \cdot 74 = 84. \]

In order to compute the variance of \( \frac{1}{3} \cdot A + \frac{1}{3} \cdot B + \frac{1}{3} \cdot C, \) observe that
\[ \text{Var}(\frac{1}{3} \cdot US + \frac{1}{3} \cdot NY + \frac{1}{3} \cdot CA) = \text{Var}(\frac{1}{3} \cdot NY + \frac{1}{3} \cdot CA), \]
as \( \frac{1}{2} A \) is a constant. It follows that
\[ \text{Var}(\frac{1}{3} \cdot US + \frac{1}{3} \cdot NY + \frac{1}{3} \cdot CA) = \text{Var}(\frac{1}{3} \cdot NY + \frac{1}{3} \cdot CA) \]
\[ = \text{Var}(\frac{1}{3} \cdot NY) + \text{Var}(\frac{1}{3} \cdot CA) + 2 \cdot \text{Cov}(\frac{1}{3} \cdot NY, \frac{1}{3} \cdot CA) \]
\[ = (1/3)^2 \text{Var}(NY) + (1/3)^2 \text{Var}(C) + 2 \cdot 1/3 \cdot 1/3 \cdot \text{Cov}(B,C) \]
\[ = (1/3)^2 \cdot 916 + (1/3)^2 \cdot 1,044 + 2 \cdot 1/3 \cdot 1/3 \cdot 348 \approx 295.11. \]
Question 3

(a) $w=0.75$:

\[
E[r_P] = E[0.75 \cdot r_A + 0.25 \cdot r_B] = 0.75 \cdot E[r_A] + 0.25 \cdot E[r_B] \\
= 0.75 \cdot 0.1 + 0.25 \cdot 0.05 = 0.0875 = 8.75\%.
\]

\[
Var(r_p) = Var(0.75 \cdot r_A + 0.25 \cdot r_B) \\
= (0.75)^2 \cdot Var(r_A) + (0.25)^2 \cdot Var(r_B) + 2 \cdot 0.75 \cdot 0.25 \cdot Corr(r_A, r_B) \cdot Std(r_A) \cdot Std(r_B) \\
= (0.75)^2 \cdot (0.2)^2 + (0.25)^2 \cdot (0.15)^2 + 2 \cdot 0.75 \cdot 0.25 \cdot 0.5 \cdot 0.2 \cdot 0.15 \approx 0.0295.
\]

\[
Std(r_p) = \sqrt{Var(r_p)} \approx \sqrt{0.0295} = 0.1718 = 17.18\%.
\]

$w=0.5$:

\[
E[r_P] = E[0.5 \cdot r_A + 0.5 \cdot r_B] = 0.5 \cdot E[r_A] + 0.5 \cdot E[r_B] \\
= 0.5 \cdot 0.1 + 0.5 \cdot 0.05 = 0.075 = 7.5\%.
\]

\[
Var(r_p) = Var(0.5 \cdot r_A + 0.5 \cdot r_B) \\
= (0.5)^2 \cdot Var(r_A) + (0.5)^2 \cdot Var(r_B) + 2 \cdot 0.5 \cdot 0.5 \cdot Corr(r_A, r_B) \cdot Std(r_A) \cdot Std(r_B) \\
= (0.5)^2 \cdot (0.2)^2 + (0.5)^2 \cdot (0.15)^2 + 2 \cdot 0.5 \cdot 0.5 \cdot 0.5 \cdot 0.2 \cdot 0.15 \approx 0.0231.
\]

\[
Std(r_p) = \sqrt{Var(r_p)} \approx \sqrt{0.0231} = 0.152 = 15.2\%.
\]

$w=0.75$:

\[
E[r_P] = E[0.25 \cdot r_A + 0.75 \cdot r_B] = 0.25 \cdot E[r_A] + 0.75 \cdot E[r_B] \\
= 0.25 \cdot 0.1 + 0.75 \cdot 0.05 = 0.0625 = 6.25\%.
\]

\[
Var(r_p) = Var(0.25 \cdot r_A + 0.75 \cdot r_B) \\
= (0.25)^2 \cdot Var(r_A) + (0.75)^2 \cdot Var(r_B) + 2 \cdot 0.25 \cdot 0.75 \cdot Corr(r_A, r_B) \cdot Std(r_A) \cdot Std(r_B) \\
= (0.25)^2 \cdot (0.2)^2 + (0.75)^2 \cdot (0.15)^2 + 2 \cdot 0.25 \cdot 0.75 \cdot 0.5 \cdot 0.2 \cdot 0.15 \approx 0.0208.
\]

\[
Std(r_p) = \sqrt{Var(r_p)} \approx \sqrt{0.0208} = 0.1422 = 14.22\%.
\]
In summary,

<table>
<thead>
<tr>
<th>Weight</th>
<th>Expected Return</th>
<th>Return Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>w=0.75</td>
<td>8.75%</td>
<td>17.18%</td>
</tr>
<tr>
<td>w=0.50</td>
<td>7.50%</td>
<td>15.20%</td>
</tr>
<tr>
<td>w=0.25</td>
<td>6.25%</td>
<td>14.22%</td>
</tr>
</tbody>
</table>
(b) The expected return of each of the three portfolios is not affected by the change in the correlation between assets A and B. It is therefore only necessary to re-compute the return standard deviation for each of the three portfolios.

\[ \text{Var}(r_p) = \text{Var}(0.75 \cdot r_A + 0.25 \cdot r_B) \]
\[ = (0.75)^2 \cdot \text{Var}(r_A) + (0.25)^2 \cdot \text{Var}(r_B) + 2 \cdot 0.75 \cdot 0.25 \cdot \text{Corr}(r_A, r_B) \cdot \text{Std}(r_A) \cdot \text{Std}(r_B) \]
\[ = (0.75)^2 \cdot 0.2^2 + (0.25)^2 \cdot 0.15^2 + 2 \cdot 0.75 \cdot 0.25 \cdot (-0.5) \cdot 0.2 \cdot 0.15 \approx 0.0183. \]

\[ \text{Std}(r_p) = \sqrt{\text{Var}(r_p)} \approx \sqrt{0.0183} \approx 0.1353 = 13.53\%. \]

\[ \text{Var}(r_p) = \text{Var}(0.5 \cdot r_A + 0.5 \cdot r_B) \]
\[ = (0.5)^2 \cdot \text{Var}(r_A) + (0.5)^2 \cdot \text{Var}(r_B) + 2 \cdot 0.5 \cdot 0.5 \cdot \text{Corr}(r_A, r_B) \cdot \text{Std}(r_A) \cdot \text{Std}(r_B) \]
\[ = (0.5)^2 \cdot 0.2^2 + (0.5)^2 \cdot 0.15^2 + 2 \cdot 0.5 \cdot 0.5 \cdot (-0.5) \cdot 0.2 \cdot 0.15 \approx 0.0081. \]

\[ \text{Std}(r_p) = \sqrt{\text{Var}(r_p)} \approx \sqrt{0.0081} = 0.09 = 9\%. \]

\[ \text{Var}(r_p) = \text{Var}(0.25 \cdot r_A + 0.75 \cdot r_B) \]
\[ = (0.25)^2 \cdot \text{Var}(r_A) + (0.75)^2 \cdot \text{Var}(r_B) + 2 \cdot 0.25 \cdot 0.75 \cdot \text{Corr}(r_A, r_B) \cdot \text{Std}(r_A) \cdot \text{Std}(r_B) \]
\[ = (0.25)^2 \cdot 0.2^2 + (0.75)^2 \cdot 0.15^2 + 2 \cdot 0.25 \cdot 0.75 \cdot (-0.5) \cdot 0.2 \cdot 0.15 \approx 0.0095. \]

\[ \text{Std}(r_p) = \sqrt{\text{Var}(r_p)} \approx \sqrt{0.0095} \approx 0.0975 = 9.75\%. \]

In summary,

<table>
<thead>
<tr>
<th>Weight</th>
<th>Expected Return</th>
<th>Return Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>w=0.75</td>
<td>8.75%</td>
<td>13.53%</td>
</tr>
<tr>
<td>w=0.50</td>
<td>7.50%</td>
<td>9.00%</td>
</tr>
<tr>
<td>w=0.25</td>
<td>6.25%</td>
<td>9.75%</td>
</tr>
</tbody>
</table>
Observe the following two properties:

- The two assets that are used to construct each of the above portfolios have standard deviation 20% and 15%. However, there are multiple portfolios whose standard deviation is lower than the standard deviation of the two building blocks. This is a manifestation of the principle of diversification.

- For each of the three portfolio weights, the return standard deviation for -0.5 correlation is strictly lower than the standard deviation for 0.5 correlation. This is a manifestation of the principle that lower correlation provides diversification benefits, which only holds as long as the portfolio weights are between 0 and 1, which they all are in this problem.